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On the boundedness of Fourier integral operators on $L^p(\mathbb{R}^n)$

Sur la continuité des opérateurs intégraux de Fourier sur $L^p(\mathbb{R}^n)$

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ABSTRACT

The aim of this Note is to present global L^p boundedness results for Fourier integral operators in $\mathbb{R}^n$. The main question is what are the decay conditions on the amplitudes for the operators to be bounded on $L^p(\mathbb{R}^n)$. Results under different sets of assumptions on phase functions and amplitudes are presented.

RÉSUMÉ

Dans cette Note nous présentons des estimations globales pour les opérateurs intégraux de Fourier dans les espaces $L^p(\mathbb{R}^n)$. Les questions d'intérêt sont les conditions des décroissance pour les amplitudes. Les résultats sont présentés sous des conditions différentes sur la fonction de phase et l'amplitude.

Version française abrégée

Dans cette Note nous occupérons des estimations globales des opérateurs integraux de Fourier sur les espaces $L^p(\mathbb{R}^n)$. Les opérateurs qui nous considérons sont de la forme

$$(\mathcal{T}u)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i[\langle x,\xi\rangle - \varphi(y,\xi)]} b(x,y,\xi)u(y)\,dy\,d\xi, \tag{1}$$

où φ est une fonction de phase à valeurs réels et b est l'amplitude. Les résultats locaux de continuité de ces opérateurs sur $L^p(\mathbb{R}^n)$ sont obtenus par Seeger, Sogge et Stein dans [21]. La continuité globale sur l'espace $L^2(\mathbb{R}^n)$ a été étudiée dans [1,3,12,17–19]. Le théorème suivant[1] extends ces résultats sur l'espace $L^p(\mathbb{R}^n)$:

Théorème 0.1. *Soit $1 < p < \infty$ et $m, \mu \in \mathbb{R}$. Supposons que l'opérateur $\mathcal{T}$ est de la forme (1), ou la fonction $\varphi \in C^\infty(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}))$ est à valeurs réels et homogène de l'ordre 1 en ξ, c'est-à-dire $\varphi(y,\tau\xi) = \tau\varphi(y,\xi)$, pour tout $\tau > 0$, $y \in \mathbb{R}^n$ et $\xi \in \mathbb{R}^n \setminus \{0\}$. En plus, supposons $|\xi| \geqslant \varepsilon > 0$ sur $\operatorname{supp} b$ et une des conditions suivantes :*

(I) *Supposons que la fonction φ pour tout $y \in \mathbb{R}^n$ et $\xi \in \mathbb{R}^n \setminus \{0\}$ satisfait les estimations*

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[1] Pour deux fonctions $f(x,y,\xi)$, $g(x,y,\xi)$, $x, y, \xi \in \mathbb{R}^n$, nous écrivons $f \prec g$ s'il existe une constante $C > 0$ de manière que, pour tout x, y, ξ, $|f(x,y,\xi)| \leqslant C|g(x,y,\xi)|$. Si $f \prec g$ et $g \prec f$, nous écrivons $f \sim g$.

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$$\left|\det \partial_y \partial_\xi \varphi(y,\xi)\right| \geqslant C > 0, \qquad \partial_y^\alpha \varphi(y,\xi) \prec \langle y\rangle^{1-|\alpha|}|\xi| \quad \textit{pour tout } \alpha,$$
$$\left\langle \nabla_\xi \varphi(y,\xi)\right\rangle \sim \langle y\rangle, \qquad \left\langle \mathrm{d}_y \varphi(y,\xi)\right\rangle \sim \langle \xi\rangle, \tag{2}$$

et

$$\partial_x^\alpha \partial_\xi^\beta \varphi(y,\xi) \prec 1$$

pour tous les multi-indices α, β *pour lesqueles* $|\alpha+\beta| \geqslant 2$. *Supposons que la fonction* $b \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ *satisfait*

$$\partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma b(x,y,\xi) \prec \langle x\rangle^{m_1} \langle y\rangle^{m_2} \langle \xi\rangle^{\mu-|\gamma|}$$

pour tout $x, y, \xi \in \mathbb{R}^n$ *et tous les multi-indices* α, β, γ, *avec* $m_1, m_2 \in \mathbb{R}$ *tels que* $m_1 + m_2 = m$.

(II) *Supposons que la fonction* φ *satisfait* (2) *sur* supp b, *et que*

$$\partial_y^\alpha \partial_\xi^\beta \varphi(y,\xi) \prec 1$$

pour tout (x,y,ξ) *dans* supp b *et tous les multi-indices* α, β *tels que* $|\alpha| \geqslant 1$ *et* $|\beta| \geqslant 1$. *En plus, supposons que* $b \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ *satisfait*

$$\partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma b(x,y,\xi) \prec \langle x\rangle^{m_1-|\alpha|} \langle y\rangle^{m_2} \langle \xi\rangle^{\mu-|\gamma|}$$

pour tout $x, y, \xi \in \mathbb{R}^n$ *et tous les multi-indices* α, β, γ, *avec* $m_1, m_2 \in \mathbb{R}$ *tels que* $m_1 + m_2 = m$.

(III) *Supposons que la fonction* φ *satisfait* (2) *sur* supp b, *et que*

$$\partial_y^\alpha \partial_\xi^\beta \varphi(y,\xi) \prec \langle y\rangle^{1-|\alpha|}$$

pour tout (x,y,ξ) *dans* supp b *et tous les multi-indices* α, β *avec* $|\beta| \geqslant 1$. *En plus, supposons que* $b \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ *satisfait*

$$\partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma b(x,y,\xi) \prec \langle x\rangle^{m_1} \langle y\rangle^{m_2-|\beta|} \langle \xi\rangle^{\mu-|\gamma|}$$

pour tout $x, y, \xi \in \mathbb{R}^n$ *et tous les multi-indices* α, β, γ, *avec* $m_1, m_2 \in \mathbb{R}$ *tels que* $m_1 + m_2 = m$.

Alors, l'operateur $\mathcal{T}$ *est borné sur* $L^p(\mathbb{R}^n)$ *si*

$$m \leqslant -n\left|\frac{1}{p} - \frac{1}{2}\right| \quad \textit{et} \quad \mu \leqslant -(n-1)\left|\frac{1}{p} - \frac{1}{2}\right|.$$

On peut démontrer un résultat plus fort pour les opérateurs intégraux de Fourier étudiée dans [9] et [17] :

Théorème 0.2. *Soit A un opérateurs intégraux de Fourier de la forme*

$$Au(x) = \int_{\mathbb{R}^n} e^{i\varphi(x,\xi)} a(x,\xi) \widehat{u}(\xi)\,\mathrm{d}\xi, \tag{3}$$

où la fonction $\varphi \in \mathcal{C}^\infty(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}))$ *est à valeurs réels et homogène de l'ordre* 1 *en* ξ, *c'est-à-dire* $\varphi(y,\tau\xi) = \tau\varphi(y,\xi)$, *pour tout* $\tau > 0$, $y \in \mathbb{R}^n$ *et* $\xi \in \mathbb{R}^n \setminus \{0\}$, *et supposons que* φ *satisfait* (2) *pour tout* $y \in \mathbb{R}^n$ *et* $\xi \in \mathbb{R}^n \setminus \{0\}$. *En plus, supposons que* $\xi \neq 0$ *sur* supp a *et*

$$\partial_x^\alpha \partial_\xi^\beta a(x,\xi) \prec \langle x\rangle^{m-|\alpha|} \langle \xi\rangle^{\mu-|\beta|},$$

pour tout $x, \xi \in \mathbb{R}^n$ *et tous multi-indices* α, β, *avec* $m, \mu \in \mathbb{R}$. *Alors, A est borné sur* $L^p(\mathbb{R}^n)$ *si*

$$m \leqslant -(n-1)\left|\frac{1}{p} - \frac{1}{2}\right| \quad \textit{et} \quad \mu \leqslant -(n-1)\left|\frac{1}{p} - \frac{1}{2}\right|. \tag{4}$$

1. Introduction

In this Note we present global $L^p(\mathbb{R}^n)$ continuity results for non-degenerate Fourier integral operators. In particular, we are interested in the question of what decay properties of the amplitude guarantee the global boundedness of Fourier integral operators from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$.

The analysis of the local L^2 boundedness of Fourier integral operators goes back to Eskin [10] and Hörmander [11], who showed that non-degenerate Fourier integral operators with amplitudes in the symbol class $S^0_{1,0}$ are locally bounded on $L^2(\mathbb{R}^n)$.

Since the 1970s this local L^2 boundedness result has been extended in different directions. The question of the global $L^2(\mathbb{R}^n)$ boundedness has been first widely investigated in the case of pseudo-differential operators. The phase is trivial in this case, so the main question is to determine minimal assumptions on the amplitude which guarantee the global $L^2(\mathbb{R}^n)$ boundedness. There are different sets of assumptions, see e.g. Calderón and Vaillancourt [4], Childs [5], Coifman and Meyer [6], Cordes [8], Sugimoto [22], etc. The question of global $L^2(\mathbb{R}^n)$ boundedness of Fourier integral operators is more subtle, and involves different sets of assumptions on both phase and amplitude. Operators arising in applications to hyperbolic equations and Feynman path integrals have been considered e.g. in Asada and Fujiwara [1], Kumano-go [12], Boulkhemair [3], Coriasco [9]. On the other hand, applications to smoothing estimates for evolution partial differential equations require less restrictive assumptions on the phase, and the necessary estimates have been established by Ruzhansky and Sugimoto [17,18].

Local L^p boundedness of Fourier integral operators has been under intensive study as well. In the case of $p \neq 2$ there is a loss of derivatives in L^p-spaces. For example, a loss of $(n-1)|1/p-1/2|$ derivatives has been established for operators appearing as solutions to the wave equations, see e.g. Beals [2], Peral [14], Miyachi [13]. Finally, Seeger, Sogge and Stein [21] showed that general non-degenerate Fourier integral operators are locally bounded in $L^p(\mathbb{R}^n)$ provided that their amplitudes are in the symbol class $S^{\mu}_{1,0}$ with $\mu \leqslant -(n-1)|1/p-1/2|$, $1<p<\infty$. In the case of $p=1$, they showed that operators of order $\mu=-(n-1)/2$ are locally bounded from the Hardy space H^1 to L^1, while Tao [23] showed that operators of the same order are also locally of weak type $(1,1)$. Extensions of these results with smaller loss of regularity under additional geometric assumptions on the canonical relations have been studied by Ruzhansky [15,16].

Here we present results on the global $L^p(\mathbb{R}^n)$ boundedness of Fourier integral operators, which depend on the growth/decay order of the amplitude in x and y variables. These results essentially extend the local L^p results of Seeger, Sogge and Stein [21] as well as global L^2 results of Asada and Fujiwara [1], and Ruzhansky and Sugimoto [17], to the global setting of $L^p(\mathbb{R}^n)$, and the main concern of this paper is the global loss of weight in the L^p-boundedness.

2. Results

Let the operator $\mathcal{T}$ be given by

$$(\mathcal{T}u)(x)=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i[\langle x,\xi\rangle-\varphi(y,\xi)]}b(x,y,\xi)u(y)\,\mathrm{d}y\,\mathrm{d}\xi, \tag{5}$$

with a real-valued phase function φ and amplitude b. The boundedness of $\mathcal{T}$ on $L^p(\mathbb{R}^n)$ holds under either of the following conditions[2] (I)–(III):

Theorem 2.1. *Let $1<p<\infty$ and $m,\mu\in\mathbb{R}$. Let $\mathcal{T}$ be operator (5), where $\varphi\in\mathcal{C}^\infty(\mathbb{R}^n\times(\mathbb{R}^n\setminus\{0\}))$ is real-valued and positively homogeneous of order 1 in ξ, i.e. $\varphi(y,\tau\xi)=\tau\varphi(y,\xi)$ for all $\tau>0$, $y\in\mathbb{R}^n$ and $\xi\in\mathbb{R}^n\setminus\{0\}$. Assume that $|\xi|\geqslant\varepsilon>0$ on $\operatorname{supp} b$ and one of the following properties:*

(I) *Let φ be such that for all $y\in\mathbb{R}^n$ and $\xi\in\mathbb{R}^n\setminus\{0\}$ we have*

$$\left|\det\partial_y\partial_\xi\varphi(y,\xi)\right|\geqslant C>0,\qquad \partial_y^\alpha\varphi(y,\xi)\prec\langle y\rangle^{1-|\alpha|}|\xi|\quad\text{for all }\alpha,$$
$$\left\langle\nabla_\xi\varphi(y,\xi)\right\rangle\sim\langle y\rangle,\qquad \left\langle \mathrm{d}_y\varphi(y,\xi)\right\rangle\sim\langle\xi\rangle, \tag{6}$$

and such that

$$\partial_x^\alpha\partial_\xi^\beta\varphi(y,\xi)\prec 1 \tag{7}$$

for all multi-indices α,β such that $|\alpha+\beta|\geqslant 2$. Let $b\in\mathcal{C}^\infty(\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n)$ satisfy

$$\partial_x^\alpha\partial_y^\beta\partial_\xi^\gamma b(x,y,\xi)\prec\langle x\rangle^{m_1}\langle y\rangle^{m_2}\langle\xi\rangle^{\mu-|\gamma|} \tag{8}$$

for all $x,y,\xi\in\mathbb{R}^n$ and all multi-indices α,β,γ, with some $m_1,m_2\in\mathbb{R}$ such that $m_1+m_2=m$.

(II) *Let φ satisfy (6) on $\operatorname{supp} b$, and*

$$\partial_y^\alpha\partial_\xi^\beta\varphi(y,\xi)\prec 1 \tag{9}$$

for all (x,y,ξ) on $\operatorname{supp} b$ and all α,β such that $|\alpha|\geqslant1$ and $|\beta|\geqslant1$, and let $b\in\mathcal{C}^\infty(\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n)$ satisfy

$$\partial_x^\alpha\partial_y^\beta\partial_\xi^\gamma b(x,y,\xi)\prec\langle x\rangle^{m_1-|\alpha|}\langle y\rangle^{m_2}\langle\xi\rangle^{\mu-|\gamma|} \tag{10}$$

for all $x,y,\xi\in\mathbb{R}^n$ and all multi-indices α,β,γ, with some $m_1,m_2\in\mathbb{R}$ such that $m_1+m_2=m$.

[2] For two functions $f(x,y,\xi)$, $g(x,y,\xi)$, $x,y,\xi\in\mathbb{R}^n$, we write $f\prec g$ if there exist a constant $C>0$ such that, for arbitrary x,y,ξ, we have $|f(x,y,\xi)|\leqslant C|g(x,y,\xi)|$. If both $f\prec g$ and $g\prec f$ hold, we write $f\sim g$.

(III) *Let* φ *satisfy* (6) *on* $\operatorname{supp} b$, *and*

$$\partial_y^\alpha \partial_\xi^\beta \varphi(y,\xi) \prec \langle y\rangle^{1-|\alpha|} \tag{11}$$

for all (x,y,ξ) *on* $\operatorname{supp} b$ *and all* α, β *such that* $|\beta| \geqslant 1$, *and let* $b \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ *satisfy*

$$\partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma b(x,y,\xi) \prec \langle x\rangle^{m_1} \langle y\rangle^{m_2-|\beta|} \langle \xi\rangle^{\mu-|\gamma|} \tag{12}$$

for all $x, y, \xi \in \mathbb{R}^n$ *and all multi-indices* α, β, γ, *with some* $m_1, m_2 \in \mathbb{R}$ *such that* $m_1 + m_2 = m$.

Then, $\mathcal{T}$ *extends to a bounded operator from* $L^p(\mathbb{R}^n)$ *to itself, provided that*

$$m \leqslant -n\left|\frac{1}{p}-\frac{1}{2}\right| \quad \text{and} \quad \mu \leqslant -(n-1)\left|\frac{1}{p}-\frac{1}{2}\right|. \tag{13}$$

We note that assumptions (6) are very natural in the sense that they ask that φ is essentially of order one in both y and ξ. Condition

$$\left|\det \partial_y \partial_\xi \varphi(y,\xi)\right| \geqslant C > 0, \tag{14}$$

for all $y \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^n \backslash \{0\}$ is simply a global version of the local graph condition of the non-degeneracy of Fourier integral operator (5). Assumption (8) says that b has a symbolic behavior in ξ and is of order $m_1 + m_2 = m$ jointly in x and y.

Assumption (II) is different from (I) in that we do not assume the boundedness (7), and assume boundedness only of mixed derivatives (i.e. $|\alpha| \geqslant 1$ and $|\beta| \geqslant 1$), but in addition assume that derivatives of b have some decay properties in (10) or in (12). In assumption (III) we also allow non-mixed derivatives (i.e. ∂_ξ^β-derivatives when $\alpha = 0$) to grow in y. Moreover, in both (II) and (III) we assume (6) to hold only on the support of b.

If the amplitude b in Theorem 2.1 is compactly supported in (x, y), Theorem 2.1 implies the local L^p boundedness under the assumptions in Seeger, Sogge and Stein [21], implying, in particular, that the order μ in Theorem 2.1 cannot be improved in general. To prove Theorem 2.1 we use interpolation between the $L^2(\mathbb{R}^n)$-boundedness and boundedness from the Hardy space $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$. The global $L^2(\mathbb{R}^n)$-bounded ness under assumptions (I) and (II)–(III) would follow from the results of Asada and Fujiwara [1] and Ruzhansky and Sugimoto [17], respectively. The boundedness from the Hardy space $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ is given as follows:

Theorem 2.2. *Let* $\mathcal{T}$ *be the Fourier integral operator* (5). *Under the hypotheses of Theorem* 2.1, *operator* $\mathcal{T}$ *extends to a bounded operator from the Hardy space* $H^1(\mathbb{R}^n)$ *to* $L^1(\mathbb{R}^n)$, *provided that* $m \leqslant -n/2$ *and* $\mu \leqslant -(n-1)/2$.

By the calculus in [19] or [20] we can establish also a result in weighted Sobolev spaces. Let $W_s^{\sigma,p}(\mathbb{R}^n)$ denote the space of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $\langle x\rangle^s (1-\Delta)^{\sigma/2} f(x)$ belongs to $L^p(\mathbb{R}^n)$.

Theorem 2.3. *Let* $1 < p < \infty$ *and let* $\sigma, s \in \mathbb{R}$. *Let* $\mathcal{T}$ *be the Fourier integral operator* (5) *as in Theorem* 2.1 *with orders* $m, \mu \in \mathbb{R}$, *and let* $m_p = n|\frac{1}{p} - \frac{1}{2}|$, $\mu_p = (n-1)|\frac{1}{p} - \frac{1}{2}|$. *Then operator* $\mathcal{T}$ *extends to a bounded operator from* $W_s^{\sigma,p}(\mathbb{R}^n)$ *to* $W_{s-m-m_p}^{\sigma-\mu-\mu_p,p}(\mathbb{R}^n)$.

We have the following better result for the Fourier integral operators studied in [9,17]:

Theorem 2.4. *Let* A *be a Fourier integral operator of the form*

$$Au(x) = \int_{\mathbb{R}^n} e^{i\varphi(x,\xi)} a(x,\xi) \widehat{u}(\xi)\, d\xi, \tag{15}$$

with a real-valued phase function $\varphi \in \mathcal{C}^\infty(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}))$ *such that* $\varphi(y, \tau\xi) = \tau\varphi(y,\xi)$ *for all* $\tau > 0$, $y \in \mathbb{R}^n$ *and* $\xi \in \mathbb{R}^n \setminus \{0\}$, *and assume that the condition* (6) *holds true for all* $y \in \mathbb{R}^n$ *and* $\xi \in \mathbb{R}^n \backslash 0$. *Moreover, assume that* $\xi \neq 0$ *on the support of the amplitude* a, *and that*

$$\partial_x^\alpha \partial_\xi^\beta a(x,\xi) \prec \langle x\rangle^{m-|\alpha|} \langle \xi\rangle^{\mu-|\beta|},$$

for all $x, \xi \in \mathbb{R}^n$ *and all multi-indices* α, β, *with some* $m, \mu \in \mathbb{R}$. *Then,* A *extends to a bounded operator from* $L^p(\mathbb{R}^n)$ *to itself, provided that*

$$m \leqslant -(n-1)\left|\frac{1}{p}-\frac{1}{2}\right| \quad \text{and} \quad \mu \leqslant -(n-1)\left|\frac{1}{p}-\frac{1}{2}\right|. \tag{16}$$

The thresholds (16) are sharp, by a modification of a counterexample described in [7] which treated the case of Fourier Lebesgue spaces. By the calculi in [9] or in [19,20] we also have:

Theorem 2.5. *Let $1 < p < \infty$ and let $\sigma, s \in \mathbb{R}$. Let A be the Fourier integral operator as in Theorem 2.4 with orders $m, \mu \in \mathbb{R}$, and let $m_p = (n-1)|\frac{1}{p} - \frac{1}{2}|$. Then operator A extends to a bounded operator from $W_s^{\sigma,p}(\mathbb{R}^n)$ to $W_{s-m-m_p}^{\sigma-\mu-m_p,p}(\mathbb{R}^n)$.*

We give a short application of the obtained results. Let $D_t = -i\partial_t$, $D_{x_j} = -i\partial_{x_j}$, and let us look at the equation

$$\begin{cases} (D_t + a(t,x,D_x))u(t,x) = 0, \quad t \neq 0, \\ u|_{t=0} = f(x), \end{cases} \tag{17}$$

where $a(t,x,\xi)$ is a real-valued classical symbol of order one depending smoothly on t, x and ξ. The strict hyperbolicity means that the principal symbol $a_1(t,x,\xi)$ is real-valued. The result of [21] states that if $f \in W^{\alpha+(n-1)|1/p-1/2|,p}(\mathbb{R}^n)$, for some $\alpha \in \mathbb{R}$, it follows that the solution satisfies $u(t,\cdot) \in W^{\alpha,p}(\mathbb{R}^n)$ locally. We now give a global version of this result. It follows from [12] that modulo smooth function, for sufficiently small times, the solution $u(t,x)$ to (17) can be constructed as a Fourier integral operator in the form (5). If we assume that a is a classical symbol such that

$$\left|\partial_t^k \partial_x^\beta \partial_\xi^\alpha a(t,x,\xi)\right| \leqslant C_{k\alpha\beta} \langle \xi \rangle^{1-|\alpha|} \tag{18}$$

holds for all $x, \xi \in \mathbb{R}^n$, all $t \in [0,T]$ for some $T > 0$, and all k, α, β, with constants $C_{k\alpha\beta}$ independent of t, x, ξ, then the phase and the amplitude of the propagator satisfy assumption (I) of Theorem 2.1. Thus, cutting off low frequencies, we obtain

Theorem 2.6. *Let the symbol $a(t,x,\xi)$ satisfy conditions (18). Let $1 < p < \infty$, and let $\chi \in C_0^\infty(\mathbb{R}^n)$ be such that $\chi(\xi) = 1$ for $|\xi| \leqslant \varepsilon$, for some $\varepsilon > 0$. If f is such that $\langle x \rangle^{n|1/p-1/2|} f(x) \in W^{(n-1)|1/p-1/2|,p}(\mathbb{R}^n)$, then for each $t \in [0,T]$, the solution $u(t,x)$ of the Cauchy problem (17) satisfies $(1-\chi(D))u(t,\cdot) \in L^p(\mathbb{R}^n)$. Moreover, for every $\alpha \in \mathbb{R}$ and $m \in \mathbb{R}$, we have the estimate*

$$\left\| \langle x \rangle^m \big(1-\chi(D)\big) u(t,\cdot) \right\|_{W^{\alpha,p}(\mathbb{R}^n)} \leqslant C_T \left\| \langle x \rangle^{m+n|1/p-1/2|} f \right\|_{W^{\alpha+(n-1)|1/p-1/2|,p}(\mathbb{R}^n)},$$

for all $t \in [0,T]$ and all f such that the right hand side norm is finite.

Theorem 2.4 can be used to obtain an analogue of this result in the SG-setting.

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