C. R. Acad. Sci. Paris, Ser. I 348 (2010) 973–976

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Partial Differential Equations

An extension of the identity **Det = det**

Une extension de l'identité **Det = det**

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ARTICLE INFO

Article history:
Received 1 April 2010
Accepted after revision 19 July 2010
Available online 12 August 2010

Presented by John Ball

ABSTRACT

In this Note we study the pointwise characterization of the distributional Jacobian of BnV maps. After recalling some basic notions, we will extend the well-known result of Müller to a more natural class of functions, using the divergence theorem to express the Jacobian as a boundary integral.

RÉSUMÉ

Dans cette Note on étudie la caractérisation ponctuelle du jacobien des applications BnV au sens des distributions. On étend un résultat bien connu de Müller à une classe plus naturelle de fonctions, en utilisant le théorème de la divergence pour écrire le jacobien comme une intégrale de contour.

1. Introduction

We first define the notion of distributional Jacobian and of BnV function:

Definition 1.1. Let $\Omega \subset \mathbb{R}^m$ be an open set, assume p and q satisfy:

$$p \geqslant n-1, \quad \frac{1}{q} + \frac{n-1}{p} \leqslant 1. \tag{1}$$

For $u \in L^q \cap W^{1,p}(\Omega, \mathbb{R}^n)$ with $m \geqslant n$, we let $j(u)$ be the $(m-n+1)$-current given by the action $\langle j(u), \omega\rangle := (-1)^n \int_\Omega u^1 \, \mathrm{d}u^2 \wedge \cdots \wedge \mathrm{d}u^n \wedge \omega$ on forms ω in $C_c^\infty(\Omega)$. The distributional Jacobian of u is the $(m-n)$-current $[Ju] := \partial j(u)$. We say that a map $u \in W^{1,p} \cap L^q$ belongs to BnV if its distributional Jacobian $[Ju]$ has finite mass (and hence it can be represented by a Radon Measure).

If $m = n$, $[Ju]$ is a distribution and a simple calculation gives that $[Ju] = \frac{1}{m} \operatorname{div}[\operatorname{Cof}(\nabla u) u]$, where $\operatorname{Cof}(\nabla u)$ is the matrix of cofactors of ∇u. This case of Definition 1.1 was first introduced by Ball in [2]. Subsequent works by Šverák [17] and Müller and Spector [15] were devoted to analyze the regularity properties of such maps and their applications to problems in elasticity. A powerful theory for these variational problems has been developed by Giaquinta, Modica and Souček (see [9]

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1631-073X/$ – see front matter
doi:10.1016/j.crma.2010.07.019

for a detailed presentation). In some relevant situations, this latter approach and the one with the distributional Jacobian are equivalent, as shown in [4] (see also [11,13,6] for further developments in this direction). The extension of the distributional Jacobian to the case $m > n$ is due to Jerrard and Soner in [12]. That paper initiated a program on the asymptotics of functionals of Ginzburg–Landau type, using the notion of BnV map. A quite thourough study of this problem has been pursued also in [1].

In this Note we prove the following two theorems:

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^m$ be an open set and let $u \in L^q \cap W^{1,p}(\Omega, \mathbb{R}^m)$ be a BnV map. Let ν be the density of the absolutely continuous part of the distributional Jacobian $[Ju]$ with respect to the Lebesgue measure: $[Ju] = \nu\mathcal{L}^m + [Ju]^s = [Ju]^a + [Ju]^s$. Then $\nu(x) = \det \nabla u(x)$ for $\mathcal{L}^m$-almost every $x \in \Omega$.*

Theorem 1.3. *If $u \in L^q \cap W^{1,p}(\Omega, \mathbb{R}^n)$ is a BnV map, then $\nu(x) = (e_1 \wedge \cdots \wedge e_m) \llcorner \mathrm{d}u^1(x) \wedge \cdots \wedge \mathrm{d}u^n(x)$ for $\mathcal{L}^m$-a.e. $x \in \Omega$ (see* [8], 1.5.2 *for the definition of $\nu \llcorner \omega$).*

Theorem 1.2 was originally proved by Müller in [14] assuming $u \in W^{1,p} \cap BnV$ with $p \geqslant n^2/(n+1)$. Müller's result was first conjectured by Ball in [2]. Note that, by Sobolev's embedding, $p \geqslant n^2/(n+1)$ implies that $u \in L^q$ for some q satisfying (1). Theorem 1.3 was claimed by the first author in [5]. Indeed, the arguments of [5] show Theorem 1.3 assuming Theorem 1.2 and are outlined here in Section 3 for completeness. However, in the aforementioned paper, the first author overlooked that Müller's proof is not valid in the full range of exponents (1).

2. Proof of Theorem 1.2

Similarly to [14], Theorem 1.2 will be proved using a blow up procedure, which needs two lemmas.

Lemma 2.1. *If $u \in BnV(B_R, \mathbb{R}^n)$ then for $\mathcal{L}^1$-a.e. $\rho \in (0, R)$:*

$$[Ju](B_\rho) = \int_{\partial B_\rho} u^1\,\mathrm{d}u^2 \wedge \cdots \wedge \mathrm{d}u^n = \int_{\partial B_\rho} \langle u^1\,\mathrm{d}u^2 \wedge \cdots \wedge \mathrm{d}u^n, \tau\rangle\,\mathrm{d}\mathcal{H}^{n-1}, \tag{2}$$

where τ is the simple $(n-1)$-vector orienting ∂B_ρ as the boundary of B_ρ.

Proof. Let $\varphi_{\delta,r}$ be a standard Lipschitz cut-off, taking the value 1 for $|x| \leqslant r - \delta$ and 0 for $|x| \geqslant r$, with $\varphi_{\delta,r}(x) = (r - |x|)/\delta$ for $r - \delta \leqslant |x| \leqslant r$. Let $f(r) := \int_{\partial B_r} u^1\,\mathrm{d}u^2 \wedge \cdots \wedge \mathrm{d}u^n$: then $f \in L^1([0,1])$ because of (1) and Fubini's Theorem. This implies that $\mathcal{L}^1$-a.e. r is a Lebesgue point, that is: $\int_{r-\delta}^{r+\delta} |f(s) - f(r)|\,\mathrm{d}s = o(\delta)$. Moreover $\langle Ju, \varphi_{\delta,r}\rangle = \langle j(u), \mathrm{d}\varphi_{\delta,r}\rangle = \int -u^1\,\mathrm{d}\varphi_{\delta,r}(x) \wedge \mathrm{d}u^2 \wedge \cdots \wedge \mathrm{d}u^n = \frac{1}{\delta}\int_{r-\delta}^{r} \mathrm{d}t \wedge \int_{\partial B_t} u^1\,\mathrm{d}u^2 \wedge \cdots \wedge \mathrm{d}u^n = \frac{1}{\delta}\int_{r-\delta}^{r} (\int_{\partial B_t} u^1\,\mathrm{d}u^2 \wedge \cdots \wedge \mathrm{d}u^n)\,\mathrm{d}\mathcal{L}^1(t)$. Hence at every Lebesgue point $\langle Ju, \varphi_{\delta,r}\rangle \to \int_{\partial B_r} u^1\,\mathrm{d}u^2 \wedge \cdots \wedge \mathrm{d}u^n$; on the other hand, by dominated convergence, $\langle [Ju], \varphi_{\delta,r}\rangle \to [Ju](B_r)$, that proves the proposition. $\square$

Definition 2.2. Let $u \in BnV(\Omega, \mathbb{R}^n)$ and let $x_0 \in B_R \subset \Omega$. We define $u_\varepsilon(y) := (u(x_0 + \varepsilon y) - u(x_0))/\varepsilon$.

Lemma 2.3. *Let u be as above and set $\delta_a(x) := a(x - x_0)$. Then $[Ju_\varepsilon] = \frac{1}{\varepsilon^n}\delta_{\frac{1}{\varepsilon}\#}[Ju]$.*

Proof. Let $\phi \in C_c^\infty(B_1)$ be a test function. Since $\langle [J(u_\varepsilon)], \phi\rangle = \langle j(u_\varepsilon), \mathrm{d}\phi\rangle$ we have:

$$\begin{aligned}
\langle [J(u_\varepsilon)], \phi\rangle &= (-1)^n \int_{B_1} \frac{u^1(x_0 + \varepsilon y) - u^1(x_0)}{\varepsilon} \det\bigl(\nabla u^2(x_0 + \varepsilon y), \ldots, \nabla u^n(x_0 + \varepsilon y), \nabla\phi(y)\bigr)\,\mathrm{d}y \\
&= (-1)^n \int_{\Omega} \frac{u^1(x) - u^1(x_0)}{\varepsilon^{n+1}} \det\left(\nabla u^2(x), \ldots, \nabla u^n(x), \nabla\phi\left(\frac{x - x_0}{\varepsilon}\right)\right)\mathrm{d}x \\
&= \frac{1}{\varepsilon^n}\left\langle j(u), \mathrm{d}\left[\phi\left(\frac{x - x_0}{\varepsilon}\right)\right]\right\rangle = \frac{1}{\varepsilon^n}\left\langle [Ju], \phi\left(\frac{x - x_0}{\varepsilon}\right)\right\rangle. \qquad \square
\end{aligned}$$

Taking the supremum over $\{\phi \in C_c^\infty(B_1)\colon \|\phi\|_\infty \leqslant 1\}$ we conclude $\|Ju_\varepsilon\| = \frac{1}{\varepsilon^n}\delta_{\frac{1}{\varepsilon}\#}\|Ju\|$. Since the Radon–Nikodym decomposition commutes with the push forward, $[Ju_\varepsilon]^a = \frac{1}{\varepsilon^n}\delta_{\frac{1}{\varepsilon}\#}[Ju]^a$ and $[Ju_\varepsilon]^s = \frac{1}{\varepsilon^n}\delta_{\frac{1}{\varepsilon}\#}[Ju]^s$, which allows to conclude

$$\bigl\|[Ju_\varepsilon]^s\bigr\|\bigl(B_r(0)\bigr) = \frac{\|[Ju]^s\|(B_{\varepsilon r}(x_0))}{\varepsilon^n} \quad \forall r > 0. \tag{3}$$

Proof of Theorem 1.2. To simplify the notation we use u_h for the function $u_{h^{-1}}$ given by Definition 2.2. We use formula (2) to the blow-up sequence (u_h) around a "good" point x_0 to get $[Ju_h](B_\rho(x_0)) = \int_{\partial B_\rho(x_0)} u_h^1 \, du_h^2 \wedge \cdots \wedge du_h^n$, and hence we let $h \uparrow \infty$ to obtain

$$\nu(x_0)|B_\rho| = \int\limits_{\partial B_\rho(x_0)} (L\cdot x)^1 L^2 \wedge \cdots \wedge L^n = \int\limits_{\partial B_\rho(x_0)} (L\cdot x)^1 \operatorname{cof}(L)^1_k \cdot \eta^k = \det(L)|B_\rho|, \tag{4}$$

where $L := \nabla u(x_0)$ and η is the exterior unit normal to ∂B_ρ.

Step 1: By the standard theory of Sobolev functions (see [7]), a.e. $x_0 \in \Omega$ satisfies the following properties:

$$(a) \qquad \lim_{r\downarrow 0} \frac{1}{r^n}\left\{ \left\| [Ju]^s \right\| \big(B_r(x_0)\big) + \int\limits_{B_r(x_0)} \big|\nu(x) - \nu(x_0)\big| \, dx \right\} = 0;$$

(b) ∇u is approximately continuous at x_0 and in particular $\int_{B_r(x_0)} |\nabla u(x) - \nabla u(x_0)|^p \, dx = o(r^n)$.

From now on we fix x_0 satisfying (a) and (b) and, without loss of generality, we assume $x_0 = 0$. Observe first of all that condition (a) and Eq. (3) imply:

$$[Ju_h]\big(B_r(0)\big) = h^n [Ju]\big(B_{\frac{r}{h}}(0)\big) = o(1) + h^n \int\limits_{B_{\frac{r}{h}}(0)} \nu(y)\, dy \to \nu(0)|B_r| \quad \forall r > 0. \tag{5}$$

Step 2: We observe that, being (u_h) a sequence, there is a set of radii $\rho \in (0,1)$ of full measure such that (2) holds for every h. Moreover by (b), using Fubini's and Fatou's Theorems, for a.e. ρ there exists a subsequence (not relabeled and possibly depending on ρ) such that $\nabla u_h \to L := \nabla u(0)$ in $L^p(\partial B_\rho)$. We fix now a radius ρ with all the properties above and we do not relabel the relevant subsequence. Hence $du_h^2 \wedge \cdots \wedge du_h^n \to L^2 \wedge \cdots \wedge L^n$ in $L^{\frac{p}{n-1}}(\partial B_\rho)$, since

$$du_h^2 \wedge \cdots \wedge du_h^n - L^2 \wedge \cdots \wedge L^n = \sum_i L^2 \wedge \cdots \wedge \big(du_k^i - L^i\big) \wedge \cdots \wedge du_k^n.$$

In the borderline case $p = (n-1)$, the convergence is improved to the first Hardy space $\mathcal{H}^1(\partial B_\rho)$ because of the Coifman–Lions–Meyer–Semmes estimate (see [3]):

$$\big\| \langle dv^2 \wedge \cdots \wedge dv^n, \tau \rangle \big\|_{\mathcal{H}^1(\partial B_\rho)} \leqslant C \big\| dv^2 \big\|_{L^{n-1}(\partial B_\rho)} \cdots \big\| dv^n \big\|_{L^{n-1}(\partial B_\rho)}. \tag{6}$$

Suppose first of all that $p > n-1$. Then by the Poincaré's inequality and the Sobolev embedding theorem, the sequence (u_h) is equicontinuous, with the estimate $\|u_h - L\cdot x - C_h\|_{C^\alpha(\partial B_\rho)} \leqslant C\|\nabla u_h - L\|_{L^p(\partial B_\rho)} \to 0$. Here C_h is the average of u_h on ∂B_ρ. Since $\int_{\partial B_\rho} du_h^2 \wedge \cdots \wedge du_h^n = 0$, we conclude,

$$[Ju_h](B_\rho) = \int\limits_{\partial B_\rho} \big(u_h^1 - C_h^1\big)\, du_h^2 \wedge \cdots \wedge du_h^n \to \int\limits_{\partial B_\rho} (L\cdot x)^1 L^2 \wedge \cdots \wedge L^n = \det(L)|B_\rho|.$$

Finally if $p = n-1$ we use the John–Nirenberg embedding and Poincaré's inequality to get $[u_h - C_h - L\cdot x]_{BMO} + \|u_h - C_h - L\cdot x\|_{L^1} \leqslant C\|\nabla u_h - L\|_{L^{n-1}(\partial B_\rho)} \to 0$. Recall that, by Fefferman's Theorem, BMO is the dual space of $\mathcal{H}^1$ and thus $|\int fg| \leqslant C([f]_{BMO} + \|f\|_{L^1})\|g\|_{\mathcal{H}^1}$ whenever fg is integrable (see [16], Chapter IV; take into account that the original Theorem of Fefferman, proved in $\mathbb{R}^n$, must be suitably modified to our situation where the domain is a compact manifold, see [10]). We thus infer that $\int_{\partial B_\rho} (u_h^1 - C_h^1)\, du_h^2 \wedge \cdots \wedge du_h^n \to \int_{\partial B_\rho} (L\cdot x)^1 L^2 \wedge \cdots \wedge L^n = \det(L)|B_\rho|$. □

3. Proof of Theorem 1.3

Given a normal current $T \in \mathbf{N}_k(\mathbb{R}^m)$ and a Lipschitz map $\pi : \mathbb{R}^m \to \mathbb{R}^l$ with $k \geqslant l$, we can define a weakly*-measurable map $x \mapsto \langle T, \pi, x\rangle \in \mathbf{N}_{k-l}(\mathbb{R}^m)$, uniquely characterized by the validity of the identity $\int_{\mathbb{R}^l} \langle T, \pi, x\rangle \psi(x)\, dx = T \llcorner (\psi \circ \pi)\, d\pi$ for every $\psi \in C_c^1(\mathbb{R}^l)$ (this is the so-called "slicing of the current", see for instance [8]). In [5], the first author proved a slicing theorem for Jacobians, namely:

Theorem 3.1. *Let $i^x : \mathbb{R}^k \to \{x\} \times \mathbb{R}^k$ be the natural injection of $\mathbb{R}^k$ into $\mathbb{R}^m$, and let $\pi : \mathbb{R}^{m-k} \times \mathbb{R}^k \to \mathbb{R}^{m-k}$ a projection, with $k \geqslant n$. Denote by u^x the trace $u(x,\cdot) = u \circ i^x$. Then $\langle [Ju], \pi, x\rangle = (-1)^{(m-k)n} i^x_\#[Ju^x]$. Moreover this property holds separately for the absolutely continuous part and the singular part of $[Ju]$.*

This theorem allows us to pass from Theorem 1.2 to Theorem 1.3.

Proof of Theorem 1.3. Set $\pi(x) = (x^1, \dots, x^{m-n})$, and $y = (x^{m-n+1}, \dots, x^n)$. By Theorem 3.1, $\langle [Ju]^a, f\,\mathrm{d}\pi\rangle = \langle [Ju]^a \llcorner \mathrm{d}\pi, f\rangle = \int_{\mathbb{R}^{m-n}} \langle [Ju]^a, \pi, x\rangle(f)\,\mathrm{d}\mathcal{L}^{m-n}(x)$. Thus, using Theorem 1.2, we conclude

$$\begin{aligned}\big\langle [Ju]^a, f\,\mathrm{d}\pi\big\rangle &= \int_{\mathbb{R}^{m-n}} \left(\int_{\mathbb{R}^n} (-1)^{(m-n)n} \det\big(\nabla_y u(x,y)\big) f(x,y)\,\mathrm{d}\mathcal{L}^n(y)\right)\mathrm{d}\mathcal{L}^{m-n}(x)\\ &= \int_{\mathbb{R}^m} \det\big(\nabla_y u(x,y)\big) f(x,y)\,\mathrm{d}y \wedge \mathrm{d}\pi = \int_{\mathbb{R}^m} f\big\langle e_1 \wedge \dots \wedge e_m \llcorner \mathrm{d}u^1 \wedge \dots \wedge \mathrm{d}u^n, \mathrm{d}\pi\big\rangle\,\mathrm{d}\mathcal{L}^m.\end{aligned}$$

It is easy to show that, for every $A \in GL(n,\mathbb{R})$, the identity $[J(u \circ A)] = \deg(A) \cdot (A^{-1}_{\#})[Ju]$ holds, where $\deg(A)$ is the sign of the determinant of A. If then I is a multiindex of length $m-n$, and $\pi^I(x) = (x^{i_1}, \dots, x^{i_{m-n}})$, we let A be a permutation matrix satisfying $\pi = \pi^I \circ A$. Then

$$\begin{aligned}\big\langle [Ju]^a, f_I\,\mathrm{d}\pi^I\big\rangle &= \deg(A) \int_{\mathbb{R}^m} f_I \circ A \big\langle e_1 \wedge \dots \wedge e_m \llcorner \mathrm{d}\big(u^1 \circ A\big) \wedge \dots \wedge \mathrm{d}\big(u^n \circ A\big), \mathrm{d}\big(\pi^I \circ A\big)\big\rangle\,\mathrm{d}\mathcal{L}^m\\ &= \deg(A) \int_{\mathbb{R}^m} A^*\big(f_I\,\mathrm{d}u^1 \wedge \dots \wedge \mathrm{d}u^n \wedge \mathrm{d}\pi^I\big) = \int_{\mathbb{R}^m} f_I\,\mathrm{d}u^1 \wedge \dots \wedge \mathrm{d}u^n \wedge \mathrm{d}\pi^I.\end{aligned}$$

It is then sufficient to write a generic form as $\omega = \sum_I f_I\,\mathrm{d}x^I$ to conclude the proof. □

Acknowledgements

The second author would like to thank Prof. Luigi Ambrosio for many useful discussions and the Universität Zürich for the generous hospitality he enjoyed during his stay. The first author is grateful to Duvan Henao for pointing out his mistake in [5].

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