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Partial Differential Equations

# $L^p$ and $W^{1,p}$ regularity of the solution of a steady transport equation

## *Régularité dans $L^p$ et $W^{1,p}$ de la solution d'une équation de transport stationnaire*

Vivette Girault [a,b], Luc Tartar [c]

[a] UPMC–Paris 6, CNRS, UMR 7598, 75005 Paris, France
[b] Department of Mathematics, TAMU, College Station, TX 77843, USA
[c] Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA 15213-3890, USA

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A B S T R A C T

We consider a steady transport system of equations in a bounded Lipschitz domain of $\mathbb{R}^d$, $2 \leqslant d \leqslant 4$, with a divergence-free transport velocity in $H^1$, tangential on the boundary. By means of two regularizations, first with a viscous penalty term and next with a Yosida approximation, we prove that an $L^p$ data, $2 \leqslant p < \infty$, yields a solution in $L^p$. We apply this result to establish that for data in $W^{1,p}$ and transport velocity in $W^{1,\infty}$, sufficiently small, the solution of a scalar transport equation belongs to $W^{1,p}$.


R É S U M É

On considère une équation de transport vectorielle stationnaire dans un domaine Lipschitz borné de $\mathbb{R}^d$, $2 \leqslant d \leqslant 4$, avec une vitesse de transport dans $H^1$, à divergence nulle, tangentielle sur le bord. A l'aide de deux régularisations, d'abord avec un terme visqueux de pénalisation et ensuite avec une approximation de Yosida, on montre que si la donnée est dans $L^p$, $2 \leqslant p < \infty$, alors la solution est dans $L^p$. On applique ce résultat pour démontrer que si la donnée d'une équation de transport scalaire est dans $W^{1,p}$ et la vitesse de transport est dans $W^{1,\infty}$, assez petite, alors la solution est dans $W^{1,p}$.


### Version française abrégée

L'étude des équations de transport dans des espaces de Banach tels que $L^p$ et $W^{1,p}$ est une partie importante de l'analyse des fluides complexes, par exemple les fluides non-Newtoniens. Soit $2 \leqslant p < \infty$ un réel. Dans un domaine $\Omega$ borné de $\mathbb{R}^d$, $2 \leqslant d \leqslant 4$, de frontière $\partial\Omega$ Lipschitz, on considère le système d'équations de transport linéaires stationnaires : pour $\boldsymbol{f} \in L^p(\Omega)^d$ donné, chercher $\boldsymbol{z} \in L^p(\Omega)^d$ vérifiant (1), avec la numérotation de la version anglaise,

$$\text{p.p. dans } \Omega, \quad \boldsymbol{C}\boldsymbol{z} + \gamma \boldsymbol{u} \cdot \nabla \boldsymbol{z} = \boldsymbol{f},$$

avec une vitesse $\boldsymbol{u} \in H^1(\Omega)^d$ à divergence nulle telle que $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ sur $\partial\Omega$ et $\boldsymbol{C} \in L^\infty(\Omega)^{d\times d}$ est uniformément défini positif, i.e. il existe $c_0 > 0$ avec

$$\text{p.p. dans } \Omega, \ \forall \boldsymbol{w} \in \mathbb{R}^d, \quad \boldsymbol{C}\boldsymbol{w} \cdot \boldsymbol{w} \geqslant c_0 |\boldsymbol{w}|^2.$$

*E-mail addresses:* girault@ann.jussieu.fr (V. Girault), tartar@cmu.edu (L. Tartar).

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D'après une extension immédiate de [5], on sait que ce système admet une et une seule solution $\boldsymbol{z} \in L^2(\Omega)^d$. Le but de la première partie de cette Note est de démontrer que $\boldsymbol{z} \in L^p(\Omega)^d$ et vérifie l'estimation (12),

$$\|\boldsymbol{z}\|_{0,p,\Omega} \leqslant \frac{1}{c_0}\|\boldsymbol{f}\|_{0,p,\Omega}.$$

Formellement, cette estimation s'obtient en testant (1) avec $|\boldsymbol{z}|^{p-2}\boldsymbol{z}$, mais ceci n'est justifié que si on sait que $\boldsymbol{z} \in L^p(\Omega)^d$. La démonstration rigoureuse procède en deux étapes :

1) On régularise (1) par pénalisation : pour $\varepsilon > 0$, on cherche $\boldsymbol{z}^\varepsilon \in H_0^1(\Omega)^d$ solution de (5),

$$\text{p.p. dans } \Omega, \quad -\varepsilon\Delta\boldsymbol{z}^\varepsilon + \boldsymbol{C}\boldsymbol{z}^\varepsilon + \gamma\boldsymbol{u}\cdot\nabla\boldsymbol{z}^\varepsilon = \boldsymbol{f}.$$

2) On effectue une régularisation de Yosida de $\boldsymbol{z}^\varepsilon$ : pour $\alpha > 0$, on cherche $\boldsymbol{\sigma}_\alpha \in L^p(\Omega)^d$ solution de (7),

$$\text{p.p. dans } \Omega, \quad \boldsymbol{\sigma}_\alpha + \alpha|\boldsymbol{\sigma}_\alpha|^{p-2}\boldsymbol{\sigma}_\alpha = \boldsymbol{z}^\varepsilon.$$

On montre que (7) a une solution unique $\boldsymbol{\sigma}_\alpha \in H_0^1(\Omega)^d$ et en testant (5) avec $|\boldsymbol{\sigma}_\alpha|^{p-2}\boldsymbol{\sigma}_\alpha \in H_0^1(\Omega)^d$, on prouve que $\boldsymbol{\sigma}_\alpha$ vérifie (12). Cette estimation uniforme permet de faire tendre $\alpha$ vers zéro dans (7) et donne la même estimation uniforme pour $\boldsymbol{z}^\varepsilon$. Ceci permet de faire tendre $\varepsilon$ vers zéro dans (5) et de montrer que $\boldsymbol{z}$ vérifie (12). À notre connaissance ce résultat pour un système est nouveau.

Dans une deuxième partie, on applique cette régularité pour démontrer que la solution $z$ de l'équation (13),

$$\text{p.p. dans } \Omega, \quad \nu z + \gamma\boldsymbol{u}\cdot\nabla z = f,$$

est dans $W^{1,p}(\Omega)$ lorsque $\Omega$ est régulier, $f \in W^{1,p}(\Omega)$ et en plus des hypothèses ci-dessus $\boldsymbol{u} \in W^{1,\infty}(\Omega)^d$, de norme assez petite :

$$|\gamma||\boldsymbol{u}|_{1,\infty,\Omega} < \nu.$$

C'est aussi vrai si $\Omega$ est Lipschitz et $\boldsymbol{u}$ s'annule au bord. Pour $2 \leqslant d \leqslant 4$, ce résultat est nouveau et améliore le Lemme 19.2 de [4], où $\Omega$ est régulier et $\boldsymbol{u} \in W^{2,r}(\Omega)^d$ avec $r > d$, ce qui permet de représenter la solution par des caractéristiques. Plus généralement, l'équation scalaire (13) a été étudiée par de nombreux auteurs, voir les contributions de [1,2] et [4], qui se placent dans un domaine régulier et utilisent une vitesse de transport régulière, et les travaux de [3] qui se placent dans tout l'espace et considèrent une vitesse de transport peu régulière.

## 1. Statement of the problem and penalty regularization

The study of transport equations in Banach spaces such as $L^p$ and $W^{1,p}$ is an important ingredient in the analysis of complex fluids, e.g. non-Newtonian fluids. Let $\Omega$ be a bounded domain in $\mathbb{R}^d$, $2 \leqslant d \leqslant 4$, with a Lipschitz-continuous boundary $\partial\Omega$ and exterior unit normal $\boldsymbol{n}$, and let $\boldsymbol{u}$ be given in

$$V_T(\Omega) = \{\boldsymbol{v} \in H^1(\Omega)^d;\ \operatorname{div}\boldsymbol{v} = 0 \text{ in } \Omega,\ \boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ on } \partial\Omega\}.$$

Consider the following system of steady transport equations: Find $\boldsymbol{z} \in L^p(\Omega)^d$ satisfying

$$\text{a.e. in } \Omega, \quad \boldsymbol{C}\boldsymbol{z} + \gamma\boldsymbol{u}\cdot\nabla\boldsymbol{z} = \boldsymbol{f}, \tag{1}$$

with the data: $2 \leqslant p < \infty$, $\gamma \neq 0$, $\boldsymbol{f} \in L^p(\Omega)^d$, and $\boldsymbol{C} \in L^\infty(\Omega)^{d\times d}$, uniformly positive definite, i.e. there exists $c_0 > 0$ such that

$$\text{a.e. in } \Omega,\ \forall\boldsymbol{w} \in \mathbb{R}^d, \quad \boldsymbol{C}\boldsymbol{w}\cdot\boldsymbol{w} \geqslant c_0|\boldsymbol{w}|^2. \tag{2}$$

Here and in the sequel, $\boldsymbol{u}\cdot\nabla\boldsymbol{v}$ denotes the product $(\nabla\boldsymbol{v})\boldsymbol{u}$ and $|\cdot|$ denotes the Euclidean norm for vectors or tensors. The scalar product of $L^2(\Omega)$ or $L^2(\Omega)^d$ is denoted indifferently by $(\cdot,\cdot)$ and the norm of $L^p(\Omega)$ is denoted by $\|\cdot\|_{0,p,\Omega}$. More generally, the seminorm and norm of $W^{1,p}(\Omega)$ are respectively denoted by $|\cdot|_{1,p,\Omega}$ and $\|\cdot\|_{1,p,\Omega}$. These notations are extended to vectors or tensors by replacing the absolute value of functions in the integral by the Euclidean norm for vectors or tensors.

It is easily shown by Galerkin's method that (1) has at least one solution $\boldsymbol{z} \in L^2(\Omega)^d$. Uniqueness is an immediate consequence of (2), and the following Green formula (cf. [5])

$$(\boldsymbol{u}\cdot\nabla\zeta, \zeta) = 0, \tag{3}$$

which holds for all $\boldsymbol{u} \in V_T(\Omega)$ and all $\zeta \in \{L^2(\Omega);\ \boldsymbol{u}\cdot\nabla\zeta \in L^2(\Omega)\}$. Then (2) and (3) imply

$$\|\boldsymbol{z}\|_{0,2,\Omega} \leqslant \frac{1}{c_0}\|\boldsymbol{f}\|_{0,2,\Omega}. \tag{4}$$

For $2 < p < \infty$ and $\boldsymbol{f} \in L^p(\Omega)^d$, we propose to establish that this solution is in $L^p(\Omega)^d$ and (4) holds with $p$ instead of 2.

Consider the following regularization of (1). For each $\varepsilon > 0$, find $\boldsymbol{z}^\varepsilon \in H_0^1(\Omega)^d$ solving:

$$\text{a.e. in } \Omega, \quad -\varepsilon\Delta\boldsymbol{z}^\varepsilon + \boldsymbol{C}\boldsymbol{z}^\varepsilon + \gamma\boldsymbol{u}\cdot\nabla\boldsymbol{z}^\varepsilon = \boldsymbol{f}. \tag{5}$$

Owing to (2) and (3), the bilinear form associated with the left-hand side of (5):

$$a^\varepsilon(\boldsymbol{z},\boldsymbol{\zeta}) = \varepsilon(\nabla\boldsymbol{z},\nabla\boldsymbol{\zeta}) + (\boldsymbol{C}\boldsymbol{z},\boldsymbol{\zeta}) + \gamma(\boldsymbol{u}\cdot\nabla\boldsymbol{z},\boldsymbol{\zeta}),$$

is elliptic and in view of the dimension, it is continuous on $H_0^1(\Omega)^d \times H_0^1(\Omega)^d$. Indeed, since $d \leqslant 4$, $H^1(\Omega) \subset L^4(\Omega)$ so that the product $(\boldsymbol{u}\cdot\nabla\boldsymbol{v})\cdot\boldsymbol{w}$ belongs to $L^1(\Omega)$ for all $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}$ in $H^1(\Omega)^d$. Therefore, by Lax–Milgram's Lemma, (5) defines a unique function $\boldsymbol{z}^\varepsilon \in H_0^1(\Omega)^d$, and

$$\|\boldsymbol{z}^\varepsilon\|_{0,2,\Omega} \leqslant \frac{1}{c_0}\|\boldsymbol{f}\|_{0,2,\Omega}, \quad \sqrt{\varepsilon}|\boldsymbol{z}^\varepsilon|_{1,2,\Omega} \leqslant \frac{1}{2\sqrt{c_0}}\|\boldsymbol{f}\|_{0,2,\Omega}. \tag{6}$$

We shall extend the first part of (6) to a uniform $L^p$ bound by a Yosida regularization. Note that we cannot test directly (5) with $|\boldsymbol{z}^\varepsilon|^{p-2}\boldsymbol{z}^\varepsilon$, because in a Lipschitz domain, this function may not be smooth enough.

## 2. Yosida regularization and $L^p$ estimate

For fixed $p$ with $2 < p < \infty$, consider the following Yosida approximation of $\boldsymbol{z}^\varepsilon$. For each $\alpha > 0$, find $\boldsymbol{\sigma}_\alpha$ in $L^p(\Omega)^d$ such that

$$\text{a.e. in } \Omega, \quad \boldsymbol{\sigma}_\alpha + \alpha|\boldsymbol{\sigma}_\alpha|^{p-2}\boldsymbol{\sigma}_\alpha = \boldsymbol{z}^\varepsilon. \tag{7}$$

**Proposition 1.** *If $\boldsymbol{z}^\varepsilon$ is given in $L^{p'}(\Omega)^d$, $1/p + 1/p' = 1$, Problem (7) has a unique solution $\boldsymbol{\sigma}_\alpha \in L^p(\Omega)^d$ and if $\boldsymbol{z}^\varepsilon \in H_0^1(\Omega)^d$, then both $\boldsymbol{\sigma}_\alpha$ and $|\boldsymbol{\sigma}_\alpha|^{p-2}\boldsymbol{\sigma}_\alpha$ belong to $H_0^1(\Omega)^d$.*

**Proof.** For any $\boldsymbol{\sigma} \in L^p(\Omega)^d$, let $\mathcal{A}(\boldsymbol{\sigma}) = \boldsymbol{\sigma} + \alpha|\boldsymbol{\sigma}|^{p-2}\boldsymbol{\sigma}$. Then $\mathcal{A}$ maps $L^p(\Omega)^d$ into $L^{p'}(\Omega)^d$; it is coercive, hemicontinuous, and monotone in $L^p(\Omega)^d$, since it satisfies

$$\forall\boldsymbol{\sigma},\boldsymbol{\tau} \in \mathbb{R}^d, \quad \big(\mathcal{A}(\boldsymbol{\sigma}) - \mathcal{A}(\boldsymbol{\tau})\big)\cdot(\boldsymbol{\sigma}-\boldsymbol{\tau}) \geqslant |\boldsymbol{\sigma}-\boldsymbol{\tau}|^2 + \alpha\big(|\boldsymbol{\sigma}|^{p-1} - |\boldsymbol{\tau}|^{p-1}\big)\big(|\boldsymbol{\sigma}| - |\boldsymbol{\tau}|\big) \geqslant |\boldsymbol{\sigma}-\boldsymbol{\tau}|^2.$$

Hence (7) has a unique solution $\boldsymbol{\sigma}_\alpha$ in $L^p(\Omega)^d$ and $\boldsymbol{\sigma}_\alpha = \varphi(\alpha|\boldsymbol{z}^\varepsilon|^{p-2})\boldsymbol{z}^\varepsilon$, where $\varphi : \mathbb{R}_+ \mapsto ]0,1]$ satisfies

$$\forall t \in \mathbb{R}_+, \quad \varphi(t)\big(1 + t\varphi(t)^{p-2}\big) = 1, \tag{8}$$

whence $\varphi(0) = 1$. Moreover, $\varphi$ is Lipschitz-continuous in $\mathbb{R}_+$ and differentiating (8), we easily derive that

$$\varphi'(t) = -\frac{\varphi(t)^{p-1}}{1 + (p-1)t\varphi(t)^{p-2}}.$$

Hence $\varphi \in \mathcal{C}^1(\mathbb{R}_+)$ and for all $t \in \mathbb{R}_+$, we have $-1 \leqslant \varphi'(t) < 0$. Now, assume that $\boldsymbol{z}^\varepsilon \in H_0^1(\Omega)^d$ and let $S^\varepsilon = \{\boldsymbol{x} \in \Omega; \boldsymbol{z}^\varepsilon(\boldsymbol{x}) \neq \boldsymbol{0}\}$. Then we have for $1 \leqslant j \leqslant d$,

$$\text{a.e. in } S^\varepsilon, \quad \frac{\partial}{\partial x_j}\big(|\boldsymbol{z}^\varepsilon|^{p-2}\big) = (p-2)|\boldsymbol{z}^\varepsilon|^{p-4}\left(\boldsymbol{z}^\varepsilon\cdot\frac{\partial\boldsymbol{z}^\varepsilon}{\partial x_j}\right).$$

Therefore

$$\text{a.e. in } S^\varepsilon, \quad \frac{\partial\boldsymbol{\sigma}_\alpha}{\partial x_j} = \varphi\big(\alpha|\boldsymbol{z}^\varepsilon|^{p-2}\big)\frac{\partial\boldsymbol{z}^\varepsilon}{\partial x_j} + \varphi'\big(\alpha|\boldsymbol{z}^\varepsilon|^{p-2}\big)\alpha(p-2)|\boldsymbol{z}^\varepsilon|^{p-4}\left(\boldsymbol{z}^\varepsilon\cdot\frac{\partial\boldsymbol{z}^\varepsilon}{\partial x_j}\right)\boldsymbol{z}^\varepsilon.$$

As $p > 2$, this formula is valid a.e. in $\Omega$ and using the previous bounds for $\varphi$ and $\varphi'$, we obtain

$$\text{a.e. in } \Omega, \quad \left|\frac{\partial\boldsymbol{\sigma}_\alpha}{\partial x_j}\right| \leqslant \big(\alpha(p-2)|\boldsymbol{z}^\varepsilon|^{p-2} + 1\big)\left|\frac{\partial\boldsymbol{z}^\varepsilon}{\partial x_j}\right|.$$

Thus $\nabla\boldsymbol{\sigma}_\alpha$ is defined a.e. in $\Omega$ and by differentiating both sides of (7), we readily derive

$$\frac{\partial\boldsymbol{z}^\varepsilon}{\partial x_j}\cdot\frac{\partial\boldsymbol{\sigma}_\alpha}{\partial x_j} = \big(1 + \alpha|\boldsymbol{\sigma}_\alpha|^{p-2}\big)\left|\frac{\partial\boldsymbol{\sigma}_\alpha}{\partial x_j}\right|^2 + \alpha(p-2)|\boldsymbol{\sigma}_\alpha|^{p-4}\left(\boldsymbol{\sigma}_\alpha\cdot\frac{\partial\boldsymbol{\sigma}_\alpha}{\partial x_j}\right)^2.$$

Hence

$$\text{a.e. in } \Omega, \quad |\nabla \boldsymbol{\sigma}_\alpha| \leqslant \frac{1}{1+\alpha|\boldsymbol{\sigma}_\alpha|^{p-2}} \left|\nabla \boldsymbol{z}^\varepsilon\right| \leqslant \left|\nabla \boldsymbol{z}^\varepsilon\right|.$$

This implies that $\boldsymbol{\sigma}_\alpha \in H^1(\Omega)^d$; then (7) yields immediately that $|\boldsymbol{\sigma}_\alpha|^{p-2}\boldsymbol{\sigma}_\alpha$ belongs also to $H^1(\Omega)^d$ and all components of both functions vanish on $\partial\Omega$. □

Therefore, we can test (5) with $|\boldsymbol{\sigma}_\alpha|^{p-2}\boldsymbol{\sigma}_\alpha$, and it gives

$$a^\varepsilon\left(\boldsymbol{z}^\varepsilon, |\boldsymbol{\sigma}_\alpha|^{p-2}\boldsymbol{\sigma}_\alpha\right) = \left(\boldsymbol{f}, |\boldsymbol{\sigma}_\alpha|^{p-2}\boldsymbol{\sigma}_\alpha\right). \tag{9}$$

**Proposition 2.** *Let $\boldsymbol{z}^\varepsilon$ be the solution of* (5) *with $\boldsymbol{f}$ given in $L^p(\Omega)^d$. Then the solution $\boldsymbol{\sigma}_\alpha$ of* (7) *satisfies the uniform bound*

$$\|\boldsymbol{\sigma}_\alpha\|_{0,p,\Omega} \leqslant \frac{1}{c_0}\|\boldsymbol{f}\|_{0,p,\Omega}. \tag{10}$$

**Proof.** Let us substitute the expression (7) for $\boldsymbol{z}^\varepsilon$ into (9). As $|\boldsymbol{\sigma}_\alpha|^{p-2}\boldsymbol{\sigma}_\alpha$ belongs to $H_0^1(\Omega)^d$, the ellipticity of $a^\varepsilon(\cdot,\cdot)$ yields

$$a^\varepsilon\left(|\boldsymbol{\sigma}_\alpha|^{p-2}\boldsymbol{\sigma}_\alpha, |\boldsymbol{\sigma}_\alpha|^{p-2}\boldsymbol{\sigma}_\alpha\right) \geqslant 0,$$

and we are left with

$$\varepsilon\left(\nabla\boldsymbol{\sigma}_\alpha, \nabla\left(|\boldsymbol{\sigma}_\alpha|^{p-2}\boldsymbol{\sigma}_\alpha\right)\right) + \gamma\left(\boldsymbol{u}\cdot\nabla\boldsymbol{\sigma}_\alpha, |\boldsymbol{\sigma}_\alpha|^{p-2}\boldsymbol{\sigma}_\alpha\right) + \left(\boldsymbol{C}\boldsymbol{\sigma}_\alpha, |\boldsymbol{\sigma}_\alpha|^{p-2}\boldsymbol{\sigma}_\alpha\right) \leqslant \|\boldsymbol{f}\|_{0,p,\Omega}\|\boldsymbol{\sigma}_\alpha\|_{0,p,\Omega}^{p-1}. \tag{11}$$

On one hand, since

$$\frac{\partial}{\partial x_j}\left(|\boldsymbol{\sigma}_\alpha|^{p-2}\boldsymbol{\sigma}_\alpha\right) = |\boldsymbol{\sigma}_\alpha|^{p-2}\frac{\partial\boldsymbol{\sigma}_\alpha}{\partial x_j} + (p-2)\boldsymbol{\sigma}_\alpha|\boldsymbol{\sigma}_\alpha|^{p-4}\left(\boldsymbol{\sigma}_\alpha\cdot\frac{\partial\boldsymbol{\sigma}_\alpha}{\partial x_j}\right),$$

we have

$$\nabla\boldsymbol{\sigma}_\alpha : \nabla\left(|\boldsymbol{\sigma}_\alpha|^{p-2}\boldsymbol{\sigma}_\alpha\right) = |\boldsymbol{\sigma}_\alpha|^{p-2}|\nabla\boldsymbol{\sigma}_\alpha|^2 + (p-2)|\boldsymbol{\sigma}_\alpha|^{p-4}\sum_{j=1}^d\left(\boldsymbol{\sigma}_\alpha\cdot\frac{\partial\boldsymbol{\sigma}_\alpha}{\partial x_j}\right)^2 \geqslant 0.$$

On the other hand, we can write

$$(\boldsymbol{u}\cdot\nabla\boldsymbol{\sigma}_\alpha)\cdot|\boldsymbol{\sigma}_\alpha|^{p-2}\boldsymbol{\sigma}_\alpha = \frac{1}{p}\boldsymbol{u}\cdot\nabla\left(|\boldsymbol{\sigma}_\alpha|^p\right).$$

Therefore, since $d \leqslant 4$, $\boldsymbol{u}\cdot\nabla(|\boldsymbol{\sigma}_\alpha|^p)$ belongs to $L^1(\Omega)$ and similarly, $|\boldsymbol{\sigma}_\alpha|^p$ belongs to $L^2(\Omega)$. As $\mathcal{D}(\overline{\Omega})$ is dense in the space (cf. [5]) $\{\zeta \in L^2(\Omega);\, \boldsymbol{u}\cdot\nabla\zeta \in L^1(\Omega)\}$, Green's formula yields

$$\int_\Omega \boldsymbol{u}\cdot\nabla\left(|\boldsymbol{\sigma}_\alpha|^p\right) = 0.$$

Hence there remains

$$c_0\|\boldsymbol{\sigma}_\alpha\|_{0,p,\Omega}^p \leqslant \left(\boldsymbol{C}\boldsymbol{\sigma}_\alpha, |\boldsymbol{\sigma}_\alpha|^{p-2}\boldsymbol{\sigma}_\alpha\right) \leqslant \|\boldsymbol{f}\|_{0,p,\Omega}\|\boldsymbol{\sigma}_\alpha\|_{0,p,\Omega}^{p-1},$$

whence (10). □

The uniform bound (10) allows to prove our main result:

**Theorem 3.** *Let $p > 2$ be a real number, and let $\Omega$, $\boldsymbol{u}$, $\gamma$, $\boldsymbol{f}$ and $\boldsymbol{C}$ be as above. Then the unique solution $\boldsymbol{z}$ of* (1) *belongs to $L^p(\Omega)^d$ and*

$$\|\boldsymbol{z}\|_{0,p,\Omega} \leqslant \frac{1}{c_0}\|\boldsymbol{f}\|_{0,p,\Omega}. \tag{12}$$

**Proof.** The uniform bound (10) shows that up to subsequences, as $\alpha$ tends to zero, $\boldsymbol{\sigma}_\alpha$ converges weakly in $L^p(\Omega)^d$ to some function $\boldsymbol{\sigma}$. In addition it implies that $|\boldsymbol{\sigma}_\alpha|^{p-2}\boldsymbol{\sigma}_\alpha$ is also uniformly bounded in $L^{p'}(\Omega)^d$. Therefore, by setting (7) into an equivalent variational form, testing it with a smooth function, and passing to the limit as $\alpha$ tends to zero, we easily derive that $\boldsymbol{\sigma} = \boldsymbol{z}^\varepsilon$. Then in view of (10),

$$\|\boldsymbol{z}^{\varepsilon}\|_{0,p,\Omega} \leqslant \frac{1}{c_0}\|\boldsymbol{f}\|_{0,p,\Omega}.$$

Hence, up to subsequences, $\boldsymbol{z}^{\varepsilon}$ converges weakly, as $\varepsilon$ tends to zero, to a function $\bar{\boldsymbol{z}}$ in $L^p(\Omega)^d$, and

$$\|\bar{\boldsymbol{z}}\|_{0,p,\Omega} \leqslant \frac{1}{c_0}\|\boldsymbol{f}\|_{0,p,\Omega}.$$

Then the two parts of (6) allow to pass to the limit in the variational formulation of (5) and we see that $\bar{\boldsymbol{z}}$ solves (1). Uniqueness of the solution of (1) completes the proof. □

## 3. Application to the $W^{1,p}$ regularity of a scalar transport equation

In the same domain $\Omega$, we consider the scalar transport equation: Find $z \in L^2(\Omega)$ satisfying

$$\text{a.e. in } \Omega, \quad \nu z + \gamma \boldsymbol{u} \cdot \nabla z = f, \tag{13}$$

where $\nu$ is a given positive constant, $\gamma \neq 0$, $f$ is given in $L^2(\Omega)$ and $\boldsymbol{u}$ in $V_T(\Omega)$. This problem has a unique solution. Several authors (for instance cf. [2] and references therein) have established that if, in addition, $\partial\Omega$ is sufficiently smooth, if $f \in H^1(\Omega)$, and if $\boldsymbol{u}$ is in $W^{1,\infty}(\Omega)^d$ with

$$|\gamma||\boldsymbol{u}|_{1,\infty,\Omega} < \nu, \tag{14}$$

then $z$ belongs to $H^1(\Omega)$ and

$$|z|_{1,2,\Omega} \leqslant \frac{1}{\nu - |\gamma||\boldsymbol{u}|_{1,\infty,\Omega}}|f|_{1,2,\Omega}. \tag{15}$$

There are several proofs of this result, but all either require a smooth boundary or rely on the $H^2$ regularity of the Laplace equation with homogeneous Dirichlet or Neumann boundary conditions. This regularity holds either if the boundary is smooth or if the domain is a convex polygon or polyhedron.

Let $p > 2$ be a real number and let $f$ be given in $W^{1,p}(\Omega)$. We shall prove that, under the above assumptions, $z \in W^{1,p}(\Omega)$. Since $z$ belongs to $H^1(\Omega)$, the gradient of each term in (13) is well defined in the sense of distributions and $\nabla z$ solves: Find $\boldsymbol{w}$ in $L^2(\Omega)^d$ such that

$$\nu \boldsymbol{w} + \gamma \boldsymbol{u} \cdot \nabla \boldsymbol{w} + \gamma (\nabla \boldsymbol{u})^T \boldsymbol{w} = \nabla f. \tag{16}$$

It is a particular case of (1) with $\boldsymbol{C} = \nu \boldsymbol{I} + \gamma(\nabla \boldsymbol{u})^T$. The fact that $\boldsymbol{u}$ belongs to $W^{1,\infty}(\Omega)^d$ implies that $\boldsymbol{C}$ is uniformly bounded in $\Omega$ and owing to (14), $\boldsymbol{C}$ satisfies (2) with $c_0 = \nu - |\gamma||\boldsymbol{u}|_{1,\infty,\Omega}$. Hence Theorem 3 implies immediately the next result.

**Theorem 4.** *Assume that $\Omega$ has a smooth boundary, or is a convex polygon or polyhedron. Let $p > 2$ be a real number, let $\nu > 0$, $\gamma \neq 0$, $f \in W^{1,p}(\Omega)$, and $\boldsymbol{u} \in V_T(\Omega) \cap W^{1,\infty}(\Omega)^d$ satisfying* (14)*. Then the unique solution $z$ of* (13) *belongs to $W^{1,p}(\Omega)$ and*

$$|z|_{1,p,\Omega} \leqslant \frac{1}{\nu - |\gamma||\boldsymbol{u}|_{1,\infty,\Omega}}|f|_{1,p,\Omega}. \tag{17}$$

**Remark 5.** The statement of Theorem 4 is valid on a bounded Lipschitz domain in the case when $\boldsymbol{u}$ vanishes on $\partial\Omega$. Indeed, it suffices to fix a smooth large ball $B$ containing $\Omega$, extend $\boldsymbol{u}$ by zero outside $\Omega$ and extend $f$ continuously in $W^{1,p}(B)$. Let $\tilde{f}$ and $\tilde{\boldsymbol{u}}$ denote the extended functions. As $\boldsymbol{u}$ vanishes on $\partial\Omega$, $\tilde{\boldsymbol{u}}$ is divergence free in $B$, it belongs to $W^{1,\infty}(B)^d$ and has the same $W^{1,\infty}$ norm as $\boldsymbol{u}$. Therefore, it satisfies (14) in $B$. Then consider the transport equation: Find $Z \in L^2(B)$ such that

$$\text{a.e. in } B, \quad \nu Z + \gamma \tilde{\boldsymbol{u}} \cdot \nabla Z = \tilde{f}.$$

The assumptions of Theorem 4 hold in $B$ and thus $Z$ belongs to $W^{1,p}(B)$. Let $z^{\star}$ denote the restriction of $Z$ to $\Omega$, and observe that

$$(\tilde{\boldsymbol{u}} \cdot \nabla Z)|_{\Omega} = \boldsymbol{u} \cdot \nabla z^{\star}.$$

Hence $z^{\star}$ is the unique solution of (13) and therefore $z = z^{\star}$. Thus $z \in W^{1,p}(\Omega)$ and the bound (17) follows immediately by testing (16) with $|\boldsymbol{w}|^{p-2}\boldsymbol{w}$ where $\boldsymbol{w} = \nabla z$.

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