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Partial Differential Equations

Existence and conservation laws for the Boltzmann–Fermi–Dirac equation in a general domain

Existence et lois de conservation pour l'équation de Boltzmann–Fermi–Dirac dans un domaine quelconque

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ABSTRACT

We prove an existence theorem for the Boltzmann–Fermi–Dirac equation for integrable collision kernels in possibly bounded domains with specular reflection at the boundaries, using the characteristic lines of the free transport. We then obtain that the solution satisfies the local conservations of mass, momentum and kinetic energy thanks to a dispersion technique.

RÉSUMÉ

On montre un théorème d'existence pour l'équation de Boltzmann–Fermi–Dirac avec un noyau de collision intégrable, dans un domaine quelconque (éventuellement borné) avec réflexion spéculaire au bord, grâce aux caractéristiques du transport libre. On obtient ensuite que la solution satisfait les conservations locale de la masse, de l'impulsion et de l'énergie cinétique, grâce à une technique de dispersion.

Version française abrégée

On considère dans cette note l'équation de Boltzmann pour les fermions. Notre but est d'établir l'existence d'une solution dans un domaine quelconque $\Omega \subset \mathbb{R}^3$ avec réflexion spéculaire au bord, et de montrer que cette solution vérifie les lois de conservation locale de la masse, du moment et de l'énergie cinétique. Cette généralisation du travail de J. Dolbeault [3] est utile dans l'étude de la limite hydrodynamique de cette équation.

On définit d'abord les caractéristiques du transport libre : $\Psi^t(x, v)$ donne la position et la vitesse au temps t d'une particule ayant au temps 0 la position x et la vitesse v, voyageant en ligne droite à vitesse constante à l'intérieur de Ω et rebondissant au bord selon la loi de la réflexion spéculaire. En conjugant la solution f avec Ψ^t, l'existence d'une telle solution se ramène à un problème de point fixe. Les lois de conservation globales de la masse et de l'énergie en découlent comme dans [3].

Pour montrer que la solution satisfait les lois de conservation locale, il faut pouvoir donner un sens à $\int_{\mathbb{R}^3} v|v|^2 f \, dv$; ceci est possible grâce à un argument de dispersion dû à B. Perthame [7] qui nous assure que si on contrôle le deuxième moment du terme source d'une équation de transport, alors le troisième moment de la solution est borné.

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1. Introduction

In this Note, we are concerned with the Boltzmann equation for a gas of fermions. It has been studied for example by J. Dolbeault [3], P.-L. Lions [2] and X. Lu [6]. Our goal is double: first, proving that the existence result and the properties of the solution proved in [3] can be extended to more general domains than $\mathbb{R}^3$, if we supplement the equation with a specular reflection condition at the boundaries; and then, establishing the local conservation laws. These two properties are needed in the study of hydrodynamic limits, especially in the incompressible inviscid regime, which is studied in a forthcoming paper [1].

Let us detail the problem. Let Ω be a subset of $\mathbb{R}^3$ having bounded curvature, and regular enough such that the function $x \mapsto n(x)$, where $n(x)$ is the outer unit normal at point x of $\partial\Omega$, can be extended into a continuously differentiable function on $\mathbb{R}^3$.

The Boltzmann equation for a gas of fermions reads

$$\partial_t f + v.\nabla_x f = Q(f) \tag{1}$$

where $f(t,x,v)$ is the density of particles which at time $t \in \mathbb{R}_+$ are at point $x \in \Omega$ with velocity $v \in \mathbb{R}^3$. It is supplemented with an initial condition

$$f(0,x,v) = f_0(x,v) \quad \forall (x,v) \in \Omega \times \mathbb{R}^3, \tag{2}$$

and a boundary condition

$$f(t,x,v) = f\big(t,x,R_x(v)\big) \quad \forall (x,v) \in \partial\Omega \times \mathbb{R}^3 \quad \text{such that } n(x).v < 0 \tag{3}$$

where $R_x(v)$ is the specular reflection law

$$R_x(v) = v - 2\big(v.n(x)\big)n(x).$$

The collision integral $Q(f)$ is given by

$$Q(f) = \int_{S^2}\int_{\mathbb{R}^3} b(v-v_*,\omega)\big(f'f'_*(1-f)(1-f_*) - ff_*\big(1-f'\big)\big(1-f'_*\big)\big)\,\mathrm{d}v_*\,\mathrm{d}\omega \tag{4}$$

with the usual notations

$$f_* = f(t,x,v_*), \qquad f' = f\big(t,x,v'\big), \qquad f'_* = f\big(t,x,v'_*\big)$$

and where the precollisional velocities (v', v'_*) are deduced from the postcollisional ones by

$$v' = v - (v-v_*).\omega\omega, \qquad v'_* = v_* + (v-v_*).\omega\omega.$$

The collision integral differs from the classical one by the terms $(1-f)$, which take into account the Pauli exclusion principle.

The function $b(w,\omega)$, known as the collision kernel, is measurable, a.e. positive, and is assumed to be in L^1 as in [3]. For the study of the conservations, we will moreover assume that

$$b(w,\omega) = q\big(|w|, |w.\omega|\big), \tag{5}$$

which is physically relevant, and endows (formally) $Q(f)$ with symmetry properties:

$$\int_{\mathbb{R}^3} \big(1, v, |v|^2\big) Q(f)\,\mathrm{d}v = 0 \tag{6}$$

for all f such that the integrals make sense. Then, integrating equation (1) against $1, v, |v|^2$, we obtain that the solution f satisfies formally the conservation of the local mass, momentum and kinetic energy:

$$\partial_t \int_{\mathbb{R}^3} \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} f\,\mathrm{d}v + \nabla_x . \int_{\mathbb{R}^3} \begin{pmatrix} v \\ v \otimes v \\ v|v|^2 \end{pmatrix} f\,\mathrm{d}v = 0. \tag{7}$$

This Note aims at showing first an existence theorem for the initial-boundary value (IBV) problem (1)–(3), and then that the local conservation laws (7) are satisfied rigorously by the solution of the problem.

2. The IBV problem

Following [5] and [4], we define the characteristic lines of the free transport equation

$$\partial_t f + v.\nabla_x f = 0$$

in the following way: for $(x, v) \in \Omega \times \mathbb{R}^3$, the characteristic line is given by $x + vt$, at least if t is small. Then, let $t_0(x, v)$ be the first value of t for which $x + t_0 v \in \partial\Omega$. Then, for $t > t_0$, the trajectory continues as $x + t_0 v + (t - t_0)R_{x+t_0 v}(v)$. If it intersects one more time the boundary of Ω at time $t_1(x, v)$, then the trajectory continues with velocity $R_{x+t_0 v+(t_1-t_0)R_{x+t_0 v}(v)}(R_{x+t_0 v}(v))$, and so on. We then define a family of maps $\{\Psi^t\}$ called the trajectory maps, $\Psi^t(x_0, v_0)$ being the point in phase space at which we arrive at time t following the trajectory line issuing from (x_0, v_0). For $t = t_0, t_1, \ldots,$ we define Ψ^t to be continuous (in time) from the right. Since Ω has bounded curvature, the trajectory intersects the boundary finitely many times in finite times. Moreover, the energy is conserved at each reflection, Ψ^t maps $\bar{\Omega} \times \mathbb{R}^3$ onto itself for every t, and the Jacobian of Ψ^t is always unity [5].

With this construction, we now define

$$f^\sharp(t, x, v) = f\big(t, \Psi^t(x, v)\big),$$

that is, we conjugate f with the free transport semigroup. This notation allows to reformulate the boundary condition in a simpler way. Indeed, notice that if $\Psi^{t-0}(x_0, v_0) = (x, v)$, then $\Psi^t(x_0, v_0) = \Psi^{t+0}(x_0, v_0) = (x, R(v))$, so that the boundary condition can be written

$$f^\sharp(t - 0, x_0, v_0) = f^\sharp(t + 0, x_0, v_0), \tag{8}$$

that is, $t \mapsto f^\sharp(t, x, v)$ is continuous.

The existence result for the IBV problem (1)–(2)–(8) is a consequence of the following lemma:

Lemma 2.1. *Let $f, h \in L^1_{\mathrm{loc}}(\mathbb{R}_+ \times \Omega \times \mathbb{R}^3)$. Then f is a solution of*

$$\partial_t f + v.\nabla_x f = h \quad \text{in } \mathcal{D}'\big(\mathbb{R}_+ \times \Omega \times \mathbb{R}^3\big) \tag{9}$$

with boundary condition (8) if and only if for almost all $(x, v) \in \Omega \times \mathbb{R}^3$, $f^\sharp$ is absolutely continuous with respect to t, $h^\sharp(t, x, v) \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ and

$$f^\sharp(t, x, v) = f_0^\sharp(t, x, v) + \int_0^t h^\sharp(s, x, v)\,\mathrm{d}s.$$

Proof. It is very similar to what is done in [4,2]. It consists un multiplying (9) by $\chi(\Psi^t(x, v))\phi(t)$ and integrating, with $\chi \in \mathcal{D}(\Omega \times \mathbb{R}^3)$ and $\phi \in \mathcal{D}(\mathbb{R}_+)$. Using then the change of variables $(X, V) = \Psi^{-t}(x, v)$ leads to

$$\int_{\Omega\times\mathbb{R}^3} \chi(x, v)\bigg(\int_{\mathbb{R}_+} \big[f^\sharp\phi'(t) + h^\sharp\phi(t)\big]\,\mathrm{d}t\bigg)\,\mathrm{d}x\,\mathrm{d}v = 0$$

and then to the announced result, since this is true for every $\chi \in \mathcal{D}(\Omega \times \mathbb{R}^3)$ and $\phi \in \mathcal{D}(\mathbb{R}_+)$. □

Equipped with this lemma, it is now possible to apply all the strategy developed in [3], and we obtain the following theorem:

Theorem 2.2. *Let Ω be either $\mathbb{R}^3$ or a regular subset of $\mathbb{R}^3$. Let the collision kernel be such that*

$$0 \leqslant b \in L^1\big(\mathbb{R}^3 \times S^2\big), \tag{10}$$

and let

$$f_0 \in L^\infty\big(\Omega \times \mathbb{R}^3\big), \quad 0 \leqslant f_0 \leqslant 1. \tag{11}$$

Then, the problem (1)–(2)–(8) has a unique solution f satisfying

$$f \in L^\infty\big(\mathbb{R}_+ \times \Omega \times \mathbb{R}^3\big), \quad 0 \leqslant f \leqslant 1 \quad \text{a.e.}$$

Moreover, f is absolutely continuous with respect to t.

Sketch of proof. The main idea is to show that the function

$$T : f \mapsto f_0\big(\Psi^{-t}(x,v)\big) + \int_0^t Q(\bar f)\big(s, \Psi^{s-t}(x,v)\big)\,\mathrm{d}s$$

has a fixed point, with

$$\bar f = \begin{cases} 0 & \text{if } f \leqslant 0, \\ f & \text{if } 0 \leqslant f \leqslant 1, \\ 1 & \text{if } f \geqslant 1. \end{cases}$$

This is achieved by showing that T is contractive in $L^\infty([0,\theta] \times \Omega \times \mathbb{R}^3)$ if θ is small enough. Then, since

$$-B \max(f, 0) \leqslant -B \bar f \leqslant Q(\bar f) \leqslant B(1 - \bar f) \leqslant B\big(1 - \min(1, f)\big)$$

where $B = \|b\|_{L^1(\mathbb{R}^3 \times S^2)}$, it comes

$$-B \max\big(f^\sharp, 0\big) \leqslant \partial_t f^\sharp \leqslant B\big(1 - \min\big(1, f^\sharp\big)\big),$$

which ensures that $f = \bar f$. It is then possible to reiterate the process on $[\theta, 2\theta]$, and so on, to construct a global solution. □

3. Conservation laws

Under the (physically relevant) assumption on the collision kernel (5) the collision operator $Q(f)$ features the very interesting symmetry properties (6), thanks to which f preserves some macroscopic quantities such as the total mass and the total kinetic energy. Indeed the following result holds as in [3] or [6]:

Theorem 3.1. *Let f_0 satisfy* (11)*, and b satisfy* (5) *and* (10)*. Then*

- *if $f_0 \in L^1(\Omega \times \mathbb{R}^3)$, then the solution f to* (1)–(2)–(8) *belongs to $C^0(\mathbb{R}_+; L^1(\Omega \times \mathbb{R}^3))$ and*

$$\iint_{\Omega \times \mathbb{R}^3} f(t,x,v)\,\mathrm{d}x\,\mathrm{d}v = \iint_{\Omega \times \mathbb{R}^3} f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v, \quad \forall t \in \mathbb{R}_+;$$

- *if $\iint_{\Omega \times \mathbb{R}^3} |v|^2 f_0\,\mathrm{d}x\,\mathrm{d}v < +\infty$, then the solution f to* (1)–(2)–(8) *is such that the function $(t,x,v) \mapsto |v|^2 f(t,x,v)$ belongs to $C^0(\mathbb{R}_+; L^1(\Omega \times \mathbb{R}^3))$ and*

$$\iint_{\Omega \times \mathbb{R}^3} |v|^2 f(t,x,v)\,\mathrm{d}x\,\mathrm{d}v = \iint_{\Omega \times \mathbb{R}^3} |v|^2 f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v, \quad \forall t \in \mathbb{R}_+.$$

Moreover, the function $(t,x,v) \mapsto |v|^2 Q(f)(t,x,v)$ belongs to $L^\infty(\mathbb{R}_+; L^1(\Omega \times \mathbb{R}^3))$.

Sketch of proof. Since the solution f to (1) belongs to $C^0(\mathbb{R}_+; L^1(\Omega \times \mathbb{R}^3))$ it is easy to see that $Q(f) \in C^0(\mathbb{R}_+; L^1(\Omega \times \mathbb{R}^3))$ and the first point is a consequence of Fubini's theorem. The second point is more tricky and is dealt with as in [3], showing first the result with a fixed point in $L^\infty([0,\theta]; L^1(\mathrm{d}x\,(1 + |v|^2)\,\mathrm{d}v))$ assuming that $\int_{\mathbb{R}^3} \int_{S^2} b(w,\omega)|w|^2\,\mathrm{d}\omega\,\mathrm{d}w < \infty$ and then using a stability argument to relax this assumption. □

We can go further and show that even the microscopic quantities are conserved. This is a consequence of the following dispersion estimate due to Perthame [7]:

Theorem 3.2. *Let $f \in L^1([0,T]; L^1(\Omega \times \mathbb{R}^3))$ be solution of*

$$\partial_t f + v.\nabla_x f = g, \qquad f(t=0) = f_0$$

with boundary condition (8)*. Assume that*

$$\iint_{\Omega \times \mathbb{R}^3} |v|^3 f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v < +\infty \quad \text{and} \quad \int_0^T \iint_{\Omega \times \mathbb{R}^3} |v|^2 g(t,x,v)\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t < C_0.$$

Then, for any bounded subset K of Ω, we have

$$\int_0^T \mathrm{d}t \int_K \mathrm{d}x \int_{\mathbb{R}^3} |v|^3 f(t,x,v)\,\mathrm{d}v \leqslant C_K.$$

This dispersion estimate is the key to obtain that f locally conserves the mass, momentum and kinetic energy:

Corollary 3.3. *Let the collision kernel b satisfy* (5) *and* (10)*. Assume that the initial data satisfy* (11)*, and*

$$\iint\limits_{\Omega\times\mathbb{R}^3} \left(1+|v|^3\right) f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v < +\infty.$$

Then, the solution f to (1)–(2)–(8) *satisfies, in distributional sense, the local conservation laws* (7)*.*

Note that because of the boundary condition (3), it is easy to see that the momentum is tangential to the boundary:

$$n.\int\limits_{\mathbb{R}^3} v f\,\mathrm{d}v = 0, \quad \forall x \in \partial\Omega.$$

Proof. In the sense of distributions, f satisfies (1). Let $\phi \in \mathcal{D}(\mathbb{R}_+ \times \Omega)$ and $\Psi_R \in \mathcal{D}(\mathbb{R}^3)$ satisfy:

$$\Psi_R(v) = \begin{cases} 1 & \text{if } |v| \leqslant R, \\ 0 & \text{if } |v| > 2R. \end{cases}$$

Then Eq. (1) implies

$$\int\limits_{\mathbb{R}_+\times\Omega} \partial_t\phi \left[\int\limits_{\mathbb{R}^3} \Psi_R(v) \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} f\,\mathrm{d}v\right] \mathrm{d}x\,\mathrm{d}t + \int\limits_{\mathbb{R}_+\times\Omega} \nabla_x\phi \left[\int\limits_{\mathbb{R}^3} \Psi_R(v) \begin{pmatrix} v \\ v\otimes v \\ v|v|^2 \end{pmatrix} f\,\mathrm{d}v\right] \mathrm{d}x\,\mathrm{d}t$$

$$= -\int\limits_{\mathbb{R}_+\times\Omega} \phi \left[\int\limits_{\mathbb{R}^3} \Psi_R(v) \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} Q(f)\,\mathrm{d}v\right] \mathrm{d}x\,\mathrm{d}t.$$

We may pass to the limit $R \to +\infty$ thanks to Theorems 3.1 and 3.2. At the end, we get the desired result. $\square$

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