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Number Theory

Perfect powers among Fibonomial coefficients

Puissances parfaites parmi les coefficients Fibonomiaux

Diego Marques [a], Alain Togbé [b]

[a] *Departamento De Matemática, Universidade De Brasília, Brasília, DF, Brazil*
[b] *Mathematics Department, Purdue University North Central, 1401 S, U.S. 421, Westville, IN 46391, USA*

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ABSTRACT

Let F_n be the nth Fibonacci number. For $1 \leqslant k \leqslant m$, let

$$\begin{bmatrix} m \\ k \end{bmatrix}_F = \frac{F_m F_{m-1} \cdots F_{m-k+1}}{F_1 \cdots F_k}$$

be the corresponding Fibonomial coefficient. In 2003, the problem of determining the perfect powers in the Fibonacci sequence was completely solved. In fact, the only solutions of $F_m = y^t$, with $m > 2$, are $(m, y, t) = (6, 2, 3)$, $(12, 12, 2)$. In this paper, we prove that the only solutions of the Diophantine equation

$$\begin{bmatrix} m \\ k \end{bmatrix}_F = y^t,$$

with $m > k + 1$ and $t > 1$, are those related to $k = 1$, that is $(m, k, y, t) = (6, 1, 2, 3)$ and $(12, 1, 12, 2)$.

RÉSUMÉ

Soit F_n le $n^{\text{ème}}$ nombre de Fibonacci. Pour $1 \leqslant k \leqslant m$, soit

$$\begin{bmatrix} m \\ k \end{bmatrix}_F = \frac{F_m F_{m-1} \cdots F_{m-k+1}}{F_1 \cdots F_k}$$

le coéfficient Fibonomial correspondant. En 2003, les puissances parfaites dans la suite de Fibonacci ont été complètement déterminées. Ainsi, les seules solutions de $F_m = y^t$, avec $m > 1$, sont $(m, y, t) = (6, 2, 3)$, $(12, 12, 2)$. Dans cet article, nous montrons que les seules solutions de l'équation diophantienne

$$\begin{bmatrix} m \\ k \end{bmatrix}_F = y^t,$$

avec $m > k + 1$ et $t > 1$, sont celles pour lesquelles $k = 1$, qui sont $(m, k, y, t) = (6, 1, 2, 3)$ et $(12, 1, 12, 2)$.

E-mail addresses: diego@mat.unb.br (D. Marques), atogbe@pnc.edu (A. Togbé).

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1. Introduction

Let $(C_n)_{n\geqslant 0}$ be a Lucas sequence given recurrently by $C_{n+2} = C_{n+1} + C_n$, for $n \geqslant 0$, where the values C_0 and C_1 are previously fixed. For instance, if $C_0 = 0$ and $C_1 = 1$, then the $C_n = F_n$ are the *Fibonacci numbers*. Also, if $C_0 = 2$ and $C_1 = 1$, the sequence $C_n = L_n$ gives the *Lucas numbers*.

The problem of finding the perfect powers in the Fibonacci sequence was a classical problem that attracted much attention during the past 40 years. In 2003, Bugeaud et al. [3, Theorem 1] confirmed the expectation: the only perfect powers in that sequence are 0, 1, 8 and 144. Such result is usually referred to the *Fibonacci Perfect Power Theorem* (FPPT) and its proof combines for the first time two powerful techniques in number theory, namely, the tools from Wiles's proof of the Last Fermat Theorem and Baker's theory on linear forms in logarithms. Furthermore, in the same paper, it was proved that the only Lucas numbers which are perfect powers are 1 and 4, see [3, Theorem 2]. In 2005, Luca and Shorey [4, Theorem 2] proved that the product of two or more consecutive Fibonacci numbers is never a perfect power except for the trivial case $F_1F_2 = 1$.

The *Fibonomial coefficient* $\begin{bmatrix} m \\ k \end{bmatrix}_F$ is defined, for $1 \leqslant k \leqslant m$, by replacing each integer appearing in the numerator and denominator of $\binom{m}{k} = \frac{m(m-1)\cdots(m-k+1)}{k(k-1)\cdots 1}$ with its respective Fibonacci number. That is

$$\begin{bmatrix} m \\ k \end{bmatrix}_F = \frac{F_m F_{m-1}\cdots F_{m-k+1}}{F_1\cdots F_k}.$$

The Fibonacci Perfect Power Theorem asserts that the solutions of the Diophantine equation $\begin{bmatrix} m \\ 1 \end{bmatrix}_F = F_m = y^t$, with $m > 2$, are $(m, y, t) = (6, 2, 3)$ and $(12, 12, 2)$. A natural question arises: what are the possible perfect powers in the sequence $\begin{bmatrix} m \\ 2 \end{bmatrix}_F$, with $m \geqslant 4$? In the sequence $\begin{bmatrix} m \\ 3 \end{bmatrix}_F$, with $m \geqslant 5$? And so on?

It is not a hard matter to prove that none of the Fibonomial coefficients, with $m-1 > k > 1$, is a Fibonacci number. Thus it would be reasonable to think that there are finitely many perfect powers in any sequence $\begin{bmatrix} m \\ k \end{bmatrix}_F$, for a fixed $k > 1$ and $m \geqslant k+2$.

In this paper, we use the Luca–Shorey method [4] to prove that the only perfect powers which appear in the Fibonomial sequence are those related to $k = 1$. Our result is the following.

Theorem 1. *The only solutions of the Diophantine equation*

$$\begin{bmatrix} m \\ k \end{bmatrix}_F = y^t \tag{1}$$

in positive integers m, k, y, t, with $m > k+1$ and $t > 1$ are $(m, k, y, t) = (6, 1, 2, 3)$ *and* $(12, 1, 12, 2)$.

In the next section, we will recall some properties related to the Fibonacci numbers that will be very useful for the proof of Theorem 1.

2. The proof

Before proceeding further, some considerations will be needed for the convenience of the reader. In fact, a *primitive divisor* p of F_n is a prime factor of F_n, which does not divide $\prod_{j=1}^{n-1} F_j$. It is known that a primitive divisor p of F_n exists whenever $n \geqslant 13$. The above statement is usually referred to the *Primitive Divisor Theorem* (see [1] and [2] for the most general version). As an application, it is immediate that if $\begin{bmatrix} m \\ k \end{bmatrix}_F = F_n$, then $\max\{m, n\} < 13$. Hence, assuming that $m-1 > k > 1$ a quick computation reveals that there are no solutions for the previous Diophantine equation in that range.

Now, we recall some interesting and helpful facts which will be essential ingredients to prove Theorem 1.

(i) $\gcd(F_m, F_n) = F_{\gcd(m,n)}$ and $F_{2n} = F_nL_n$.

(ii) Let p be a prime number and let ρ_p be the smallest positive index n such that p divides F_n (called *rank of apparition of* p). Then $F_n \equiv 0 \pmod p$ if and only if $n \equiv 0 \pmod{\rho_p}$ and $p \equiv (5/p) \pmod{\rho_p}$ (see [7]). Here $(5/p)$ is the usual Legendre symbol.

(iii) If $d = \gcd(m, n)$, then

$$\gcd(F_m, L_n) = \begin{cases} L_d, & \text{if } m/d \text{ is even and } n/d \text{ is odd;} \\ 1 \text{ or } 2, & \text{otherwise.} \end{cases}$$

(iv) (Sylvester Theorem [6]) If n and k are positive integers, with $n > k$, then the product of k consecutive integers

$$\prod_{n,k} = n(n+1)\cdots(n+k-1)$$

is necessarily divisible by a prime $p > k$ (i.e., $P(\prod_{n,k}) > k$, where $P(m)$ denotes the greatest prime divisor of a positive integer m).

Let $[a,b]$ denote the set $\{a, a+1, \ldots, b\}$, where a, b are integers such that $a < b$. Now, we are ready to deal with the proof of Theorem 1.

By FPPT, the only perfect powers in the Fibonacci sequence are $F_0 = 0$, $F_1 = F_2 = 1$, $F_6 = 8$, and $F_{12} = 144$ giving the solutions for our Diophantine equation in the case $k = 1$. So, we suppose that $k > 1$. We can rewrite Eq. (1) into the form

$$F_m \cdots F_{m-k+1} = y^t F_1 \cdots F_k. \tag{2}$$

Moreover, we can assume that t is a prime number. Using computational tools, one can see that for all $\ell \in [1, 190]$, there exists a prime number $p > 17$, such that p^2 does not divide F_ℓ. Suppose that $m \in [13, 190]$, then by the Primitive Divisor Theorem, F_m has a primitive divisor p. By Eq. (2), p must divide y, since $t \geqslant 2$, then p^2 divides F_m but this gives a contradiction. So, we consider $m > 190$. We will split our proof in two cases.

Case 1. $m \leqslant 2k - 1$.

We claim that there exists $i \in [0, k-1]$, such that $m - i$ is a power of 2. In fact, if $m = 2k - 1$, then $[m - k + 1, m] = [k, 2k - 1]$, and when $m \leqslant 2k - 2$, we have

$$I = \left(\frac{m}{2}, m\right] \subseteq [m - k + 1, m]$$

and thus the interval I contains a unique power of 2, say $m - i = 2^\mu$ (in fact, each interval $(x, 2x]$, with $x > 0$, contains a unique power of 2). Thus, if $i \neq j \in [0, k-1]$, then $\operatorname{ord}_2(m - j) < \mu$. Note that $2k - 1 \geqslant m \geqslant 191$ and then $k \geqslant 96$. Since $2^\mu > m/2 \geqslant k/2 \geqslant 48$, we get $\mu \geqslant 6$. Using item (i), we rewrite Eq. (2) into the form

$$F_{2^{\mu-1}} L_{2^{\mu-1}} \prod_{\substack{j \in [0, k-1] \\ j \neq i}} F_{m-j} = y^t F_1 \cdots F_k. \tag{3}$$

As $\gcd(L_{2^{\mu-1}}, F_j) = 1$ for $j \in [1, k]$, $\gcd(L_{2^{\mu-1}}, F_{m-j}) = 1$, for $i \neq j \in [0, k-1]$ we get $\gcd(L_{2^{\mu-1}}, F_{2^{\mu-1}}) = 1$ or 2. However F_m is even iff $3|m$ and then $\gcd(L_{2^{\mu-1}}, F_{2^{\mu-1}}) = 1$. Thus, Eq. (3) leads to $L_{2^{\mu-1}} = y_1^t$, for some integer $y_1 > 1$. Since $2^{\mu-1} \geqslant 32$, then $L_{2^{\mu-1}}$ cannot be a perfect power, see [3, Theorem 2], completing the proof in this case.

Case 2. $m > 2k - 1$ and so $m - k + 1 > k$.

Since $m, m-1, \ldots, m-k+1$ are k consecutive numbers greater than k, we get by Sylvester Theorem, that $Q = P(m(m-1)\cdots(m-k+1)) > k$. It follows that $Q \geqslant 5$. Indeed, suppose that $Q = 2, 3$. If $Q = 2$, then $k = 1$, which is impossible as we suppose that $k > 1$. If $Q = 3$, then $k = 1, 2$. We need only to consider the case $k = 2$. In this case, we have $3 = P(m(m-1))$ and $m(m-1) = 2^a 3^b$. Since $\gcd(m, m-1) = 1$, then one can see that $m = 3^b$ and $m - 1 = 2^a$ or $m = 2^a$ and $m - 1 = 3^b$. These systems give the equation $2^a - 3^b = \pm 1$. We know that the only solution of this equation is $(a, b) = (3, 2)$, see [5, p. 178, (3.1)]. Thus we have $m \leqslant 3^2 < 190$, which is impossible. Therefore $Q \geqslant 5$.

Since there are exactly k consecutive numbers in the sequence $m, m-1, \ldots, m-k+1$, we must have that Q divides a unique $m - j$, for some $j \in [0, k-1]$. Write $m - j = Q_1 t$, where $Q_1 = Q^\mu$ and $\gcd(Q, t) = 1$. So, we can rewrite Eq. (2) into the form

$$F_{Q_1} \left(\frac{F_{m-j}}{F_{Q_1}}\right) \prod_{\substack{i \in [0, k-1] \\ i \neq j}} F_{m-i} = y^t F_1 \cdots F_k.$$

Observe that $\gcd(F_{Q_1}, F_{m-i}) = 1$ and $\gcd(F_{Q_1}, F_j) = 1$ because $\operatorname{ord}_Q(m - i) = 1$, for $i \neq j$ and $j < k < Q$. Also, we have $\gcd(F_{Q_1}, F_{m-j}/F_{Q_1}) = \gcd(F_{Q_1}, t) = 1$. To prove this last equality, we use (ii) to conclude that if p is a prime factor of F_{Q_1}, then $\rho_p = Q^a$, with $a \in [1, \mu]$ and $p \geqslant 2\rho_p - 1$, because ρ_p is odd (and then 2 divides $p \pm 1$). Thus

$$p \geqslant 2\rho_p - 1 = 2Q^a - 1 \geqslant 2Q - 1 > Q > P(m - j) > P(t).$$

So $\gcd(F_{Q_1}, t) = 1$. Therefore, for some y_1 (factor of y) we have $F_{Q_1} = y_1^t$. By FPPT, we infer that Q_1 is either 6 or 12 (keep in mind that $Q_1 > 5$). However this is impossible because $Q_1 = Q^\mu$ and Q is a prime number.

Hence, we must only to consider the range $2 \leqslant k \leqslant 10$ and $k + 2 \leqslant m \leqslant 12$. We wrote a simple program in Mathematica to see that no value helps to get a perfect power. We recall that for being a perfect power, the greatest common divisor of the exponents needs to be > 2 but this does not happen. Indeed, all number N in this sequence possess a prime factor $p < 17$, such that p^2 does not divide N. Thus we have our desired result.

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