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Ordinary Differential Equations

On summability of formal solutions to a Cauchy problem and generalization of Mordell's Theorem

Sur la sommabilité des solutions formelles d'un problème de Cauchy et la généralisation d'un théorème de Mordell

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ABSTRACT

In this Note, we shall consider the heat equation with a singular initial condition $\varphi(z) = \frac{1}{1-e^z}$, $z \in \mathbb{C} \setminus 2\pi i\mathbb{Z}$. The aim is to establish relations among three sums of a divergent formal solution to this Cauchy problem: Borel-sum based on known results in Lutz et al. (1999) [4] and two q-Borel sums obtained by means of Heat Kernel and Theta function respectively in Zhang (1999) [8] and in Ramis and Zhang (2002) [7], Zhang (2002) [9]. It is shown that the Borel-sum is equal to q-Borel sum given by the integral expression with Heat Kernel and that the relation between two q-Borel sums gives rise to a natural generalization of Mordell's Theorem.

RÉSUMÉ

Dans cette Note, nous considérons l'équation de la chaleur avec une condition initiale singulière $\varphi(z) = \frac{1}{1-e^z}$, $z \in \mathbb{C} \setminus 2\pi i\mathbb{Z}$. Le but est d'établir des relations entre trois sommes d'une solution formelle divergente de ce problème de Cauchy: la somme de Borel basée sur des résultats de Lutz et al. (1999) [4] et deux sommes de Borel q-analogues obtenues respectivement au moyen du noyau de la chaleur (Zhang, 1999) [8] et la fonction Theta (Ramis and Zhang, 2002; Zhang, 2002) [7,9]. Nous montrons que la somme de Borel est égale à la somme q-Borel donnée par le noyau de la chaleur et que la relation entre les deux sommes q-Borel donne lieu à une généralisation naturelle d'un théorème de Mordell.

Version française abrégée

Les équations du type réaction–diffusion font partie des équations aux dérivées partielles possédant des propriétés fort similaires aux équations différentielles ordinaires en matière d'étude des singularités dans un plan complexe. Depuis le travail [4], on sait sous quelles conditions un problème de Cauchy associé à l'équation de la chaleur admet une solution série entière divergente mais sommable au sens de Borel (*cf.* [1,5]). D'habitude, deux fonctions classiques apparaissent parmi les solutions « classiques » à la diffusion de la chaleur : le noyau de la chaleur et la fonction Theta de Jacobi. Ces deux

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fonctions spéciales vérifient aussi une même équation fonctionnelle dite « aux q-différences ». Dans les travaux [8] et [7,9], elles ont été utilisées comme des fonctions q-exponentielles dans deux procédés différents de sommation qui reprennent, tous les deux, un mécanisme analogue de la méthode classique de Borel–Laplace. Le présent travail consiste à comprendre les liens entre le modèle de l'équation de la chaleur et celui de l'equation aux q-différences.

Plus précisemment, nous considérons le problème de Cauchy :

$$\partial_\tau u - \partial_z^2 u = 0, \qquad u(0,z) = \varphi(z) = \frac{1}{1-e^z},$$

avec $(\tau,z) \in \mathbb{C}\times\mathbb{C}$. Notons $\hat{u}(\tau,z)$ la série formelle solution du problème ; on a : $\hat{u}(\tau,z) = \sum_{n=0}^{\infty} \frac{\varphi^{(2n)}(z)}{n!}\tau^n$, qui est divergent en τ and holomorphe en $z \in \mathbf{C} \setminus 2\pi i\mathbb{Z}$. D'autre part, si l'on pose $q = e^{-\tau}$ et $x = e^z$, le probème de Cauchy admet la solution formelle $\sum_{n=0}^{\infty} e^{n^2\tau+nz}$, laquelle vérifie également l'équation aux q-différences

$$\frac{x}{q} y\left(\frac{x}{q^2}\right) - y(x) = -1.$$

Si $\Re(\tau) > 0$, on a $0 < |q| < 1$ et la série $\hat{v}(x,q) = \sum_{n=0}^{\infty} q^{-n^2}x^n$ sera divergente pour tout $x \in \mathbb{C}^*$.

Examinons les différentes sommes associées aux séries $\hat{u}(\tau,z)$ et $\hat{v}(x,q)$ introduites ci-dessus. Soit $k \in \mathbb{Z}$ et appelons $U_k(\tau,z)$ la somme de Borel de $\hat{u}(\tau,z)$ dans la direction $\mathbb{R}^+$ relative à la variable τ ; elle est définie par la relation (3) pour $\Re(\tau) > 0$ et $z \in \Omega_k = \{z \in \mathbb{C} \mid 2k\pi < \Im(z) < 2(k+1)\pi\}$.

Si $\alpha \in \mathbb{R} \setminus 2\pi\mathbb{Z}$, notons $f_\alpha(x,q)$ la q-Borel somme de $\hat{v}(x,q)$ donnée par l'expression (5) à l'aide du noyau de la chaleur. Pour tout $\lambda \in \mathbb{C}^* \setminus q^{2\mathbb{Z}}$, notons $g_\lambda(x,q)$ la q-Borel somme de $\hat{v}(x,q)$ définie avec la fonction Theta par (6). Soit enfin $F_\alpha(\tau,z)$ et $G_\lambda(\tau,z)$ la fonction respective associée à $f_\alpha(x,q)$ et $g_\lambda(x,q)$ via le changement de variable donné par (7).

Théorème.

(i) *Pour tout $k \in \mathbb{Z}$, on a la relation $U_k(\tau,z) = F_\alpha(\tau,z)$ pourvu que $\alpha \in (2k\pi, 2(k+1)\pi)$, $\Re(\tau) > 0$ et $z \in \Omega_k$.*
(ii) *Pour tout $\alpha \in (-2\pi, 0)$, $\lambda \in \mathbb{C}^* \setminus q^{2\mathbb{Z}}$ et $x \in \mathbb{C}^* \setminus (-\lambda q^{2\mathbb{Z}+1})$, on a :*

$$f_\alpha(x,q) = g_\lambda(x,q) - i\sqrt{\frac{\pi}{\log 1/q}}\, e^{\frac{(\log x)^2}{4\log q}}\, g_{\lambda^*}\big(x^*, q^*\big),$$

où les q^*, x^* et λ^* désignent les variables modulaires *définies par*

$$q^* = e^{\pi^2/\log q}, \qquad x^* = e^{-\pi i \frac{\log x}{\log q}}, \qquad \lambda^* = e^{-\pi i \frac{\log\lambda}{\log q}}.$$

L'assertion (ii) ci-dessus illustre à quel point les variables modulaires sont commodes pour représenter le phénomène de Stokes. Elle contient un résultat de Mordell [6] comme cas particulier. En effet, le noyau de la chaleur, comme la fonction Theta de Jacobi, est souvent utilisé dans l'étude du nombre de classes de formes quadratiques, tout en étant lié à la théorie des fonctions doublement périodiques.

1. Introduction and main results

From formal solutions of ordinary or partial differential equations, one may give different solutions by different summation processes ([1,3–5], ...). This phenomenon occurs for functional equations such as difference or q-difference equations ([2,7–9], ...). Our motivation is to reveal the relationship existing between different sums of one power series which is related to the heat equation and which may also be viewed as solution to a singular q-difference equation.

More precisely, let us consider the following Cauchy problem for the complex heat equation

$$\partial_\tau u - \partial_z^2 u = 0, \qquad u(0,z) = \varphi(z) = \frac{1}{1-e^z}, \tag{1}$$

where $(\tau,z) \in \mathbb{C}\times\mathbb{C}$, and $\varphi(z)$ is defined on $\mathbb{C} \setminus 2\pi i\mathbb{Z}$. It has a unique formal power series solution $\hat{u}(\tau,z) := \sum_{n=0}^{\infty} \frac{\varphi^{(2n)}(z)}{n!}\tau^n$, which is divergent in τ and holomorphic in z. On the other hand, if we consider the power series expansion $\hat{\varphi}(z) := \sum_{n=0}^{\infty} e^{nz}$ of $\varphi(z)$ in e^z, then (1) has a formal solution $\sum_{n=0}^{\infty} e^{n^2\tau+nz}$, which satisfies the q-difference equation

$$\frac{x}{q} y\left(\frac{x}{q^2}\right) - y(x) = -1, \tag{2}$$

provided that $q = e^{-\tau}$ and $x = e^z$. If $\Re(\tau) > 0$, then $0 < |q| < 1$ and the q-series $\hat{v}(x,q) := \sum_{n=0}^{\infty} q^{-n^2}x^n$ is divergent for all $x \in \mathbb{C}^*$. Utilizing the results on singular q-difference equation from [8] and [7,9], one can give two different q-Borel sums of $\hat{v}(x,q)$: $f_\alpha(x,q)$ and $g_\lambda(x,q)$, by which one shall get two sums of $\hat{u}(\tau,z)$, denoted by $F_\alpha(\tau,z)$ and $G_\lambda(\tau,z)$ (see (7)).

Following the classical Borel summation method with respect to τ, one can obtain the Borel-sum of the power series $\hat{u}(\tau, z)$ (*cf.* [4]). Here, we only give the expression of the Borel-sum $U_k(\tau, z)$ in the direction of the positive real axis, *i.e.*, for any $k \in \mathbb{Z}$,

$$U_k(\tau, z) = \frac{1}{\sqrt{4\pi\tau}} \int_{-\infty}^{+\infty} \frac{e^{-\frac{s^2}{4\tau}}}{1 - e^{z+s}}\, ds, \tag{3}$$

where $\Re(\tau) > 0$ and $z \in \Omega_k := \{z \in \mathbb{C} \mid 2k\pi < \Im(z) < 2(k+1)\pi\}$. Then our main results are summarized as follows:

Theorem 1.1. *For any given $k \in \mathbb{Z}$, we have $U_k(\tau, z) = F_\alpha(\tau, z)$ where $\alpha \in (2k\pi, 2(k+1)\pi)$, $\Re(\tau) > 0$ and $z \in \Omega_k$.*

Theorem 1.2. *The following relation holds for all $\alpha \in (-2\pi, 0)$, $\lambda \in \mathbb{C}^* \setminus q^{2\mathbb{Z}}$ and $x \in \mathbb{C}^* \setminus (-\lambda q^{2\mathbb{Z}+1})$:*

$$f_\alpha(x, q) = g_\lambda(x, q) - i\sqrt{\frac{\pi}{\log 1/q}}\, e^{\frac{(\log x)^2}{4\log q}}\, g_{\lambda^*}\big(x^*, q^*\big),$$

where q^*, x^* and λ^* are the modular variables defined by

$$q^* = e^{\pi^2/\log q}, \qquad x^* = e^{-\pi i \frac{\log x}{\log q}}, \qquad \lambda^* = e^{-\pi i \frac{\log \lambda}{\log q}}.$$

Replacing λ by $\frac{1}{q}e^{\pi i}$ in Theorem 1.2, one can get the following Mordell's result in [6]:

Corollary 1.3 *(Mordell's Theorem). Let f be the integral function of x defined by*

$$f(x, \omega) = -i \sum_{m\ odd}^{\pm\infty} \frac{(-1)^{\frac{1}{2}(m-1)} q^{\frac{1}{4}m^2} e^{m\pi i x}}{1 + q^m},$$

and let θ_{11} be the following Jacobi's Theta function:

$$\theta_{11}(x, \omega) = -i \sum_{m\ odd}^{\pm\infty} (-1)^{\frac{1}{2}(m-1)} q^{\frac{1}{4}m^2} e^{m\pi i x}.$$

Then

$$\int_{-\infty}^{\infty} \frac{e^{\pi i \omega t^2 - 2\pi t x}}{e^{2\pi t} - 1}\, dt = \frac{f(\frac{x}{\omega}, -\frac{1}{\omega}) + i\omega f(x, \omega)}{\omega\theta_{11}(x, \omega)},$$

where the path of integration is taken as a straight line parallel to the real axis of t and below it at a distance less than unity.

Remark 1.4. Theorem 1.2 is a natural generalization of Mordell's Theorem on the class number of the definite binary quadratics; see [6] for a historic exposition of this study.

The rest of this Note is organized as follows. In Section 2, we shall show how to derive Borel-sum $U_k(\tau, z)$ and two q-Borel sums $F_\alpha(\tau, z)$ and $G_\lambda(\tau, z)$. In Section 3, a sketch of proofs of our main results will be given.

2. Sums of formal solutions

2.1. Classical Borel–Laplace summation

For definitions of Borel transform, Laplace transform and Borel summability, see [1,5].

Since the Cauchy data $\varphi(z)$ is not analytic at the origin, one cannot directly apply results in [4] to get the Borel-sum of $\hat{u}(\tau, z)$ in the direction $\mathbb{R}^+$. But using variable transformation $y = z - (2k+1)\pi i$, one can transfer the unlimited strip-type domain Ω_k into the domain $\tilde{\Omega} := \{y \in \mathbb{C} \mid -\pi < \Im(y) < \pi\}$, which contains the origin $y = 0$. Hence, problem (1) has the equivalent form:

$$\partial_\tau u - \partial_y^2 u = 0, \qquad u(0, y) = \frac{1}{1 - e^{y+(2k+1)\pi i}}. \tag{4}$$

The initial condition $u(0, y)$ is clearly analytic on $\tilde{\Omega}$. By Theorem 3.3 in [4], problem (4) has a solution $u_k(\tau, y)$ defined by

$$u_k(\tau, y) = \frac{1}{\sqrt{4\pi\tau}} \int_{-\infty}^{+\infty} e^{-\frac{s^2}{4\tau}} \frac{1}{1 - e^{y+s+(2k+1)\pi i}}\, ds,$$

where $\Re(\tau) > 0$ and $y \in \tilde{\Omega}$.

Therefore, one can get a solution of (1):

$$U_k(\tau, z) = u_k\big(\tau, z - (2k+1)\pi i\big) = \frac{1}{\sqrt{4\pi\tau}} \int_{-\infty}^{+\infty} e^{-\frac{s^2}{4\tau}} \frac{1}{1 - e^{z+s}}\, ds,$$

where $\Re(\tau) > 0$ and $z \in \Omega_k$.

Remark 2.1. In fact, we can derive the Borel-sum of $\hat{u}(\tau, z)$ in the direction of any argument θ with $-\pi < \theta < \pi$ by a similar approach. But notice that then the variable τ belongs to $\{\tau \in \mathbb{C} \mid |\arg\tau - \theta| < \pi/2\}$ and that z belongs to an oblique strip-type domain as follows:

$$\Omega_k^{\theta/2} := \left\{ z \in \mathbb{C} \;\middle|\; 2k\pi < \Im(z) - \tan\left(\frac{\theta}{2}\right)\Re(z) < 2(k+1)\pi \right\}$$

where k denotes any given integer.

2.2. Sums by means of Heat Kernel and Theta function

Here, to the q-series $\hat{v}(x, q)$ will be associated two families of sums: one involves Heat Kernel, the other uses Jacobi's Theta function.

Proposition 2.2. *For any $\alpha \in \mathbb{R} \setminus 2\pi\mathbb{Z}$, if we set*

$$f_\alpha(x, q) = \frac{1}{\sqrt{4\pi \log 1/q}} \int_{-\infty+\alpha i}^{\infty+\alpha i} \frac{e^{\frac{(\log x - \xi)^2}{4\log q}}}{1 - e^{\xi}}\, d\xi, \tag{5}$$

then we have the following properties.

(i) *The function $x \mapsto f_\alpha(x, q)$ is holomorphic over the Riemann surface of the logarithm $\tilde{\mathbb{C}}^*$, and if α and β belong to a common interval of the set $\mathbb{R} \setminus (2\pi\mathbb{Z})$, then $f_\alpha(x, q) = f_\beta(x, q)$.*

(ii) $f_\alpha(xe^{-2\pi i}, q) - f_\alpha(x, q) = i\sqrt{\frac{\pi}{\log 1/q}} e^{\frac{(\log x - 2\pi k_\alpha i)^2}{4\log q}}$*, where we denote by k_α the unique integer such that $2\pi k_\alpha \in (\alpha, \alpha + 2\pi)$.*

(iii) *Moreover, $f_\alpha(x, q)$ is the* unique *solution of* (2) *which admits $\sum_{n=0}^{\infty} q^{-n^2} x^n$ as q-Gevrey asymptotic expansion at $x = 0$ along the direction $\mathbb{R}^+ e^{\alpha i}$.*

For more information in this direction, see [8]. Let us denote by $\theta(x, q)$ the following Jacobi Theta function: $\theta(x, q) = \sum_{n\in\mathbb{Z}} q^{n^2} x^n$ for $0 < |q| < 1$ and $x \in \mathbb{C}$.

Proposition 2.3. *Let $\lambda \in \mathbb{C}^* \setminus q^{2\mathbb{Z}}$ and $x \in \mathbb{C}^* \setminus (-\lambda q^{2\mathbb{Z}+1})$ and consider*

$$g_\lambda(x, q) = \frac{1}{\theta(\frac{\lambda}{x}, q)} \sum_{n\in\mathbb{Z}} \frac{q^{n^2}}{1 - \lambda q^{2n}} \left(\frac{\lambda}{x}\right)^n. \tag{6}$$

Then we have the following properties.

(i) *The function $x \mapsto g_\lambda(x, q)$ is holomorphic over $\mathbb{C}^* \setminus (-\lambda q^{2\mathbb{Z}+1})$ and admits $(-\lambda q^{2\mathbb{Z}+1})$ as a set of simple poles.*

(ii) $g_\lambda(xe^{2\pi i}, q) = g_\lambda(x, q)$.

(iii) *Moreover, $g_\lambda(x, q)$ is the* unique *solution of* (2) *satisfying the property* (i) *in the above and admitting the power series $\sum_{n=0}^{\infty} q^{-n^2} x^n$ as asymptotic expansion in the following sense: for $x \to 0$ in $\mathbb{C}^* \setminus (-\lambda q^{2\mathbb{Z}+1})$, there exist $C > 0$, $A > 0$ such that for all $N \in \mathbb{N}^*$ and $\epsilon > 0$ small enough*

$$\left| g_\lambda(x, q) - \sum_{n=0}^{N-1} q^{-n^2} x^n \right| \leqslant \frac{C}{\epsilon} A^N q^{-N^2} |x|^N, \quad \forall x \in \mathbb{C}^* \setminus \bigcup_{n\in\mathbb{Z}} \{ q^{2n-1} x \mid |x + \lambda| \leqslant \epsilon \}.$$

See [7,9] for (i), (ii) and the asymptotic expansion property stated in (iii). The uniqueness can be obtained from a standard argument employed in the elliptic function theory. The above propositions give rise to two different sums of $\hat{u}(\tau, z)$ as follows:

$$F_\alpha(\tau, z) := f_\alpha\big(e^z, e^{-\tau}\big), \qquad G_\lambda(\tau, z) := g_\lambda\big(e^z, e^{-\tau}\big). \tag{7}$$

3. The sketch of proofs of main results

Proof of Theorem 1.1. For fixed k and for all $z \in \Omega_k$, one has

$$U_k(\tau, z) = \frac{1}{\sqrt{4\pi\tau}} \int_{-\infty}^{+\infty} e^{-\frac{s^2}{4\tau}} \frac{1}{1 - e^{z+s}}\, ds = \frac{1}{\sqrt{4\pi\tau}} \int_{-\infty+\Im(z)i}^{+\infty+\Im(z)i} e^{-\frac{(\xi-z)^2}{4\tau}} \frac{1}{1 - e^{\xi}}\, d\xi.$$

We know that putting different $z \in \Omega_k$, the integral path would change, but the integrand has no singularity along any $(-\infty + \Im(z)i, +\infty + \Im(z)i)$. So by Proposition 2.2(i), one concludes that $U_k(\tau, z) = F_\alpha(\tau, z)$ for any $\alpha \in (2k\pi, 2(k+1)\pi)$. It proves our wanted theorem. □

Remark 3.1. By Proposition 2.2(ii) and Theorem 1.1, one can get the analytic continuation of $U_k(\tau, z)$ and

$$U_{k+1}(\tau, z) - U_k(\tau, z - 2\pi i) = i\sqrt{\frac{\pi}{\tau}} e^{-\frac{[z-2(k+1)\pi i]^2}{4\tau}}$$

for $z \in \Omega_{k+1}$.

Proof of Theorem 1.2. To prove the result, we use the simple fact that if y_1 and y_2 are two solutions of (2), then $y_1 - y_2$ will be a solution of the associated homogeneous equation and be infinitely flat. Let us consider

$$h_\lambda(x, q) := \frac{1}{i}\sqrt{\frac{\log 1/q}{\pi}} e^{-\frac{(\log x)^2}{4\log q}} \big(f_\alpha(x, q) - g_\lambda(x, q)\big),$$

where $\alpha \in (-2\pi, 0)$, $\lambda \in \mathbb{C}^* \setminus q^{2\mathbb{Z}}$ and $x \in \mathbb{C}^* \setminus (-\lambda q^{2\mathbb{Z}+1})$.

Thanks to Propositions 2.2 and 2.3, one can find the following relations:

$$\begin{cases} h_\lambda\left(\dfrac{x}{q^2}, q\right) = h_\lambda(x, q), \\ e^{-\frac{\pi i}{\log q}\log x} e^{-\frac{\pi^2}{\log q}} h_\lambda\big(x e^{-2\pi i}, q\big) - h_\lambda(x, q) = 1. \end{cases}$$

If we set $q^* = e^{\frac{\pi^2}{\log q}}$, $x^* = e^{-\pi i \frac{\log x}{\log q}}$ and $h^*_\lambda(x^*, q) = -h_\lambda(x, q)$, it follows that

$$\begin{cases} h^*_\lambda\big(x^* e^{2\pi i}, q\big) = h^*_\lambda\big(x^*, q\big), \\ \dfrac{x^*}{q^*} h^*_\lambda\left(\dfrac{x^*}{q^{*2}}, q\right) - h^*_\lambda\big(x^*, q\big) = -1. \end{cases}$$

By observing that $h^*_\lambda(x^*, q)$ is holomorphic over $\mathbb{C}^* \setminus (-\lambda^* q^{*2\mathbb{Z}+1})$ and admits simple poles on the q-spiral $(-\lambda^* q^{*2\mathbb{Z}+1})$, one concludes that $h^*_\lambda(x^*, q) = g_{\lambda^*}(x^*, q^*)$. This completes the proof of Theorem 1.2. □

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