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Sharp Carleman estimates for singular parabolic equations and application to Lipschitz stability in inverse source problems

Inégalités fines de Carleman pour des problèmes paraboliques singuliers et application à des problèmes inverses

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ABSTRACT

We address the question of Lipschitz stability results in inverse source problems for the heat equation perturbed by a singular inverse-square potential $-\mu/|x|^2$ when $\mu \leqslant \mu^\star(N) := (N-2)^2/4$ where $\mu^\star(N)$ is the optimal constant in the so-called Hardy inequality. Following Immanuvilov and Yamamoto (1998) [9], our proof is based on Carleman inequalities like those developed by Fursikov and Immanuvilov (1996) [8] for the classical heat equation. However, we need here to take into account the singularity. Therefore, the first step of the proof consists in some improvements of the Carleman inequalities specifically developed for equations with inverse-square potentials by Vancostenoble and Zuazua (2008) [15] and next Ervedoza (2008) [7]. Major steps rely on various improved forms of the Hardy inequality.

RÉSUMÉ

On étudie la stabilité Lipschitzienne pour des problèmes inverses de détermination d'une source pour l'équation de la chaleur perturbée par un potentiel singulier de la forme $-\mu/|x|^2$ avec $\mu \leqslant \mu^\star(N) := (N-2)^2/4$ où $\mu^\star(N)$ est la constante optimale de l'inégalité de Hardy. Suivant Immanuvilov et Yamamoto (1998) [9], notre preuve repose sur des inégalités de Carleman telles que celles introduites par Fursikov et Immanuvilov (1996) [8] pour l'équation de la chaleur classique. Cependant, il faut ici tenir compte de la singularité. La première étape de la preuve consiste donc en une amélioration des inegalités de Carleman spécifiquement démontrées pour des equations avec un potentiel singulier par Vancostenoble et Zuazua (2008) [15] puis Ervedoza (2008) [7]. Certaines étapes majeures reposent sur diverses formes améliorées de l'inégalité de Hardy.

Version française abrégée

Nous présentons dans cette note des résultats de stabilité Lipschitzienne pour des problèmes inverses de détermination d'une source pour l'équation de la chaleur perturbée par un potentiel singulier de la forme $-\mu/|x|^2$. D'après [1], il y a existence globale de solutions positives lorsque $\mu \leqslant \mu^\star(N)$ et au contraire blow-up instantané lorsque $\mu > \mu^\star(N)$, la constante

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$\mu^\star(N) := (N-2)^2/4$ étant celle de l'inégalité de Hardy (1). La contrôlabilité à zéro de telles équations a récemment été étudiée dans [15,7] via des estimations de Carleman spécifiques basées sur un choix approprié des fonctions poids tenant compte de la singularité. On s'intéresse ici à des problèmes inverses pour de telles équations. L'objectif est de déterminer un terme source à partir de certaines mesures de la solution mais sans la donnée de la condition initiale. Pour l'équation de la chaleur classique, ce problème a été étudié par Imanuvilov–Yamamoto [9] à l'aide des inégalités de Carleman de [8]. Nous montrons ici un résultat de stabilité Lipschtzienne qui étend celui de [9] au cas de l'équation de la chaleur avec un potentiel singulier du type $-\mu/|x|^2$ lorsque $\mu \leqslant \mu^\star(N)$.

Soient $N \in \mathbb{N}$ et $\Omega \subset \mathbb{R}^N$ un ouvert borné tel que $0 \in \Omega$ et de frontière Γ de classe $\mathcal{C}^2$. Soient également $\omega \subset \Omega$ un ouvert non vide et $T > 0$ donnés. On note $\Omega_T = \Omega \times (0,T)$, $\Gamma_T = \Gamma \times (0,T)$ et $\omega_T = \omega \times (0,T)$. Ensuite, on fixe t_0 tel que $0 < t_0 < T$, on introduit $T_0 := (t_0+T)/2$ et on note $\Omega_{t_0,T} = \Omega \times (t_0,T)$ et $\omega_{t_0,T} = \omega \times (t_0,T)$. L'objectif est de déterminer le terme source $g(x,t)$ dans (2) à partir de la connaissance de $y|_{\omega_{t_0,T}}$ et $y(\cdot,T_0)$, mais sans connaître la donnée initiale y_0.

Pour tout $\mu \leqslant \mu^\star(N)$, le cadre fonctionnel naturellement associé au problème (2), i.e. l'espace de Hilbert $H_\mu(\Omega)$ et l'opérateur non borné $(A_\mu, D(A_\mu))$ (qui engendre un semi-groupe analytique de contractions dans $L^2(\Omega)$, voir [17]) sont décrits dans la Section 2.

Pour l'étude du problème inverse, l'étape essentielle réside dans l'obtention d'inégalités de Carleman suffisamment fines pour le système (2) améliorant celles de [15,7]. On note $\mathcal{B} = B(0,1)$, $\mathcal{B}_T = \mathcal{B} \times (0,T)$, $\tilde{\mathcal{O}} = \Omega \setminus \overline{\mathcal{B}}$, and $\tilde{\mathcal{O}}_T = \tilde{\mathcal{O}} \times (0,T)$. On introduit la fonction σ définie par (3) où s et λ sont des paramètres positifs et Ψ, θ et Φ des fonctions décrites dans la Section 3. On obtient alors le résultat suivant :

Théorème 0.1. *On suppose $\mu \leqslant \mu^\star(N)$. Soient $0 < \gamma < 2$ et $k = 1 + 2/\gamma$. Il existe $\lambda_0 \geqslant 1$ tel que, pour tout $\lambda \geqslant \lambda_0$, il existe $s_0(\lambda) > 0$ et $C = C(\Omega, \omega, T, \gamma) > 0$, indépendante de $\mu \leqslant \mu^\star(N)$, telle que pour tout $s \geqslant s_0(\lambda)$, les solutions y de (2) satisfont (4).*

Le Théorème 0.1 améliore les inégalités [15,7] dans deux directions.

- Tout d'abord, [7] donne une estimation suffisante de y^2 dans $\tilde{\mathcal{O}}_T$ mais pas dans $\mathcal{B}_T$. En particulier, pour obtenir une estimation suffisante (et uniforme par rapport au paramètre $\mu \leqslant \mu^\star(N)$), on ne suit pas le choix de [7] qui fixe $k = 3$. Au contraire, on laisse, comme dans [15], le choix de k libre ce qui autorise une estimation plus fine. Un outil essentiel à ce niveau est l'inégalité de Hardy améliorée (5).
- Ensuite, [7] ne donne aucune estimation sur y_t^2 ce qui est pourtant utile pour conclure si l'on suit la méthode de [9]. Mais estimer y_t^2 nécessite tout d'abord une estimation de $|x|^2|\nabla y|^2$ dans Ω_T, qui n'est pas donnée dans [7] excepté dans $\tilde{\mathcal{O}}_T$. Notons qu'il est impossible d'avoir une estimation uniforme par rapport à $\mu \leqslant \mu^\star(N)$ de $|\nabla y|^2$ dans tout Ω_T puisque y n'appartient a priori pas à $H_0^1(\Omega)$ dans le cas critique $\mu = \mu^\star(N)$. L'idée est donc d'obtenir une estimation uniforme de la quantité $|\nabla y|^2 - \mu y^2/|x|^2$, puis de conclure en utilisant l'inégalité de Hardy spécifique (6) démontrée dans [16].

Comme application au problème inverse considéré, on obtient, en prenant les sources g dans les espaces $\mathcal{G}(C_0)$ définis par (7) :

Théorème 0.2. *Soient $T > 0$, $t_0 \in (0,T)$, $T_0 = (t_0+T)/2$ et ω un ouvert non vide de Ω. On suppose que $\mu \leqslant \mu^\star(N)$ et $y_0 \in L^2(\Omega)$. Alors, pour tout $C_0 > 0$, il existe $C = C(\Omega, \omega, t_0, T, C_0) > 0$, constante independante de $\mu \leqslant \mu^\star(N)$, telle que, pour tout $g \in \mathcal{G}(C_0)$, les solutions y de (2) satisfont (8).*

Un cas particulier intéressant de problème inverse est le cas où g est de la forme $g(x,t) = f(x)r(x,t)$ où $r \in \mathcal{C}^1(\overline{\Omega} \times [0,T])$ est donnée et où $f \in L^2(\Omega)$ est l'inconnue. On obtient alors le résultat de stabilité du Théorème 4.2 et par conséquent le résultat d'unicité qui suit.

1. Introduction

In this work, we establish Lipschitz stability in inverse source problems for heat equations with singular *inverse-square* potentials of the form $-\mu/|x|^2$ arising, e.g., in the context of combustion theory and quantum mechanics. By [1], global existence for positive solutions holds if $\mu \leqslant \mu^\star(N)$ versus instantaneous blow-up when $\mu > \mu^\star(N)$. Here $\mu^\star(N) := (N-2)^2/4$ is the optimal constant in the Hardy inequality:

$$\forall z \in H_0^1(\Omega), \quad \mu^\star(N) \int_\Omega \frac{z^2}{|x|^2}\,\mathrm{d}x \leqslant \int_\Omega |\nabla z|^2\,\mathrm{d}x. \tag{1}$$

Well-posedness (even in the critical case $\mu = \mu^\star(N)$) has been precisely studied in [17]. Recently, the null-controllability properties of such equations have been addressed in [15,7] via specific Carleman estimates based on a suitable choice of the weights taking into account the singularity (and related to the case of degenerate heat equations [3,12]). In the present paper, we consider some inverse source problems for such heat equations with inverse-square potential. We are interested in determining a source term from the knowledge of some measurements of the solution but without any knowledge of the initial condition of the system. For the standard heat equation, this problem has been studied by Imanuvilov and Yamamoto [9] using the Carleman estimates of [8]. (See also [13] for the case of the wave equation.) Our result consists in an unconditional and global Lipschitz stability estimate that extends [9] to the case of the heat equations with inverse-square potential when $\mu \leqslant \mu^\star(N)$. The key ingredient consists in sharp Carleman estimates for the N-d heat equation with inverse-square potential that improved those derived in [15,7]. Major steps in the proof of this new Carleman inequalities are based on some improved forms of Hardy inequalities.

2. Statement of the problem, functional setting and well-posedness

Let $N \in \mathbb{N}$ be given and consider a smooth bounded domain $\Omega \subset \mathbb{R}^N$ such that $0 \in \Omega$ and whose boundary Γ is of class $\mathcal{C}^2$. We also consider some non-empty open set $\omega \subset \Omega$ and some $T > 0$. We denote $\Omega_T = \Omega \times (0, T)$, $\Gamma_T = \Gamma \times (0, T)$ and $\omega_T = \omega \times (0, T)$. Next we consider t_0 such that $0 < t_0 < T$, we introduce $T_0 := (t_0 + T)/2$ and we denote $\Omega_{t_0,T} = \Omega \times (t_0, T)$ and $\omega_{t_0,T} = \omega \times (t_0, T)$. We are interested in determining the source term $g(x,t)$ in

$$\begin{cases} y_t - \Delta y - \dfrac{\mu}{|x|^2} y = g(x,t) & (x,t) \in \Omega_T, \\ y(x,t) = 0 & (x,t) \in \Gamma_T, \\ y(x,0) = y_0(x) & x \in \Omega. \end{cases} \tag{2}$$

This inverse problem consists in retrieving g from the knowledge of the local trace $y_{|\omega_{t_0,T}}$ and some slice $y(\cdot, T_0)$, but without any knowledge of the initial condition y_0 of the system. Before going any further, let us recall the functional setting associated to problem (2). First of all, for all $\mu \leqslant \mu^\star(N)$, we introduce the Hilbert space $H_\mu(\Omega)$ obtained as the completion of $H^1_0(\Omega)$ with respect to the norm

$$\forall z \in H^1_0(\Omega), \quad \|z\|_{H_\mu(\Omega)} := \left(\int_\Omega \left(|\nabla z|^2 - \frac{\mu}{|x|^2} z^2 \right) \mathrm{d}x \right)^{1/2}.$$

In the sub-critical case $\mu < \mu^\star(N)$ and when $N \neq 2$, $H_\mu(\Omega) = H^1_0(\Omega)$, whereas, in the critical case $\mu = \mu^\star(N)$, then the space $H_{\mu^\star(N)}(\Omega)$ is strictly (even slightly) larger than $H^1_0(\Omega)$, see [17].

Next, for all $\mu \leqslant \mu^\star(N)$, we define the following unbounded operator:

$$D(A_\mu) := \left\{ z \in H_\mu(\Omega) \cap H^2_{\mathrm{loc}}(\Omega) \;\middle|\; \Delta z + \frac{\mu}{|x|^2} z \in L^2(\Omega) \right\}, \quad \forall z \in D(A_\mu),\ A_\mu z := -\Delta z - \frac{\mu}{|x|^2} z.$$

We also denote $\|z\|_{D(A_\mu)} := \|A_\mu z\|_{L^2(\Omega)}$. Then A_μ generates an analytic semi-group of contractions in $L^2(\Omega)$, see [17]. Therefore well-posedness and regularity follow from standard theory of semi-groups.

3. Sharp Carleman estimates for parabolic equations with inverse-square potential

Let us denote $\mathcal{B} = B(0,1)$, $\mathcal{B}_T = \mathcal{B} \times (0, T)$, $\tilde{\mathcal{O}} = \Omega \setminus \overline{\mathcal{B}}$, and $\tilde{\mathcal{O}}_T = \tilde{\mathcal{O}} \times (0, T)$. A major step to establish Carleman inequalities relies in a suitable choice of the weight functions: as in [7], we introduce

$$\sigma(x,t) = s\theta(t)\left(e^{2\lambda\|\Psi\|_\infty} - \frac{1}{2}|x|^2 - e^{\lambda\Psi(x)} \right) \tag{3}$$

where s and λ are positive parameters, Ψ is a smooth function satisfying $\Psi(x) = \ln(|x|)$ for $x \in \mathcal{B}$, $\Psi(x) = 0$ for $x \in \Gamma$ and $\Psi(x) > 0$ for $x \in \tilde{\mathcal{O}}$ and such that there exists some open set ω_0 and $\delta > 0$ such that $\overline{\omega_0} \subset \omega$ and $|\nabla\Psi(x)| \geqslant \delta$ for $x \in \overline{\Omega} \setminus \omega_0$. See [7] for the construction of Ψ. Finally, θ is taken as in [3,15], in the form $\theta(t) = 1/(t(T-t))^k$ with $k > 0$ to be chosen. Finally, we introduce $\Phi(x) = e^{\lambda\Psi(x)}$ and following the ideas of [15,7], we prove:

Theorem 3.1. *Assume $\mu \leqslant \mu^\star(N)$. Consider $0 < \gamma < 2$ and choose $k = 1 + 2/\gamma$. There exists $\lambda_0 \geqslant 1$ such that, for all $\lambda \geqslant \lambda_0$, there exist $s_0(\lambda) > 0$ and $C = C(\Omega, \omega, T, \gamma) > 0$, independent of $\mu \leqslant \mu^\star(N)$, such that for all $s \geqslant s_0(\lambda)$, the solutions y of (2) satisfy*

$$\begin{aligned} & s^3 \int_{\Omega_T} \theta^3 e^{-2\sigma} |x|^2 y^2 \,\mathrm{d}x\,\mathrm{d}t + s^3 \int_{\tilde{\mathcal{O}}_T} \theta^3 \Phi^3 e^{-2\sigma} y^2 \,\mathrm{d}x\,\mathrm{d}t + s \int_{\Omega_T} \theta e^{-2\sigma} \frac{y^2}{|x|^\gamma} \,\mathrm{d}x\,\mathrm{d}t + s \int_{\tilde{\mathcal{O}}_T} \theta \Phi e^{-2\sigma} |\nabla y|^2 \,\mathrm{d}x\,\mathrm{d}t \\ & \quad + \int_{\Omega_T} e^{-2\sigma} \left(|\nabla y|^2 - \frac{\mu}{|x|^2} y^2 \right) \mathrm{d}x\,\mathrm{d}t + \frac{1}{s} \int_{\tilde{\mathcal{O}}_T} \frac{1}{\theta\Phi} e^{-2\sigma} y_t^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{1}{s} \int_{\mathcal{B}_T} \frac{1}{\theta} e^{-2\sigma} y_t^2 \,\mathrm{d}x\,\mathrm{d}t \\ & \quad \leqslant C s^2 e^{4\lambda\|\Psi\|_\infty} \int_{\Omega_T} e^{-2\sigma} g^2 \,\mathrm{d}x\,\mathrm{d}t + C s^5 \lambda^4 e^{4\lambda\|\Psi\|_\infty} \int_{\omega_T} \theta^3 \Phi^3 e^{-\sigma} y^2 \,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{4}$$

This improves the inequalities derived in the 1-d case in [15] and in the N-d case in [7]. Indeed, for our purpose, it was necessary to the inequality in [7] in two directions.

First of all, [7] only provides a good estimate of y^2 in $\tilde{\mathcal{O}}_T$. But the estimates given in $\mathcal{B}_T$ are not sufficient to conclude. In particular, this would only allow to provide a result for any fixed $\mu < \mu^\star(N)$ and the result would not uniform with respect to $\mu \leqslant \mu^\star(N)$. In [7], the choice $k = 3$ is made. Here we prefer, as in [15] to let the choice of k free: this allows to obtain a sharper estimate, in particular some uniform estimate of $y^2/|x|^\gamma$ for any γ such that $1 < \gamma < 2$ that is useful to conclude. In this purpose, we use some ideas of [15] and in particular the following improved form of Hardy inequality:

Lemma 3.2. *For all $n \geqslant 0$ and for all $0 < \gamma < 2$, there exists $K_0 = K_0(\gamma, n) > 0$ such that*

$$\forall z \in H_0^1(\Omega), \quad \mu^\star(N) \int_\Omega \frac{z^2}{|x|^2}\,dx + n \int_\Omega \frac{z^2}{|x|^\gamma}\,dx \leqslant \int_\Omega |\nabla z|^2\,dx + K_0 \int_\Omega z^2\,dx. \tag{5}$$

Secondly, [7] does not provide any estimate of y_t^2 that is however useful to conclude if we follow the method of [9]. But estimating y_t^2 requires to first estimate of $|x|^2|\nabla y|^2$ in Ω_T, which is not made in [7] except in $\tilde{\mathcal{O}}_T$. Observe that one cannot expect to obtain any estimate of $|\nabla y|^2$ in the whole domain Ω_T that would be uniform with respect to $\mu \leqslant \mu^\star(N)$ since y a priori does not belong to $H_0^1(\Omega)$ in the critical case. The idea here is to derive an estimate of the quantity $|\nabla y|^2 - \mu y^2/|x|^2$ instead of $|\nabla y|^2$. For this quantity, it is possible to get some *uniform* estimate. Next we conclude by using the following form of Hardy inequality:

Lemma 3.3. *(See [16, Theorem 1.1].) Denoting $R_\Omega := \max_{x\in\Omega} |x|$, we have*

$$\forall z \in H_0^1(\Omega), \quad \int_\Omega |x|^2|\nabla z|^2\,dx \leqslant R_\Omega^2 \int_\Omega \left(|\nabla z|^2 - \mu^\star(N)\frac{z^2}{|x|^2}\right)dx + \frac{N^2-4}{4}\int_\Omega z^2\,dx. \tag{6}$$

4. Lipschitz stability results in inverse source problems

Let us we introduce the spaces in which the source terms g will be taken: for any $C_0 > 0$,

$$\mathcal{G}(C_0) := \left\{ g \in H^1\big(0, T; L^2(\Omega)\big) \;\middle|\; \left|\frac{\partial g}{\partial t}(x,t)\right| \leqslant C_0\big|g(x, T_0)\big| \text{ a.e. } (x,t) \in (0,T) \times \overline{\Omega} \right\}. \tag{7}$$

Following [9] and using Theorem 3.1, we deduce our main result:

Theorem 4.1. *Let $T > 0$, $t_0 \in (0, T)$, $T_0 = (t_0 + T)/2$ be given and ω some non-empty open set of Ω. Assume that $\mu \leqslant \mu^\star(N)$ and $y_0 \in L^2(\Omega)$. Then, for all $C_0 > 0$, there exists $C = C(\Omega, \omega, t_0, T, C_0) > 0$, independent of $\mu \leqslant \mu^\star(N)$, such that, for all $g \in \mathcal{G}(C_0)$, the solutions y of* (2) *satisfy*

$$\|g\|^2_{L^2(\Omega_T)} \leqslant C\left\|\Delta y(\cdot, T_0) + \frac{\mu}{|x|^2}y(\cdot, T_0)\right\|^2_{L^2(\Omega)} + C\|y_t\|^2_{L^2(\omega_{t_0,T})}. \tag{8}$$

Compared to the result of [9], the difference relies in the fact that $\|g\|_{L^2(\Omega_T)}$ is estimated by the $D(A_\mu)$-norm of $y(\cdot, T_0)$ instead of the $H^2(\Omega)$-norm as it was the case for the classical heat equation considered in [9]. Let us recall that the assumption on g, i.e. $g \in \mathcal{G}(C_0)$ for some $C_0 > 0$, is necessary. Otherwise, in general, non-uniqueness occurs, see, e.g. [10, p. 159].

Observe that the constant C in Theorem 4.1 remains bounded with respect to $\mu \leqslant \mu^\star(N)$ and thus the result still holds true for the critical parameter $\mu = \mu^\star(N)$. This comes from the fact that we managed to derive sharp Carleman inequalities with suitable uniform estimates with respect to $\mu \leqslant \mu^\star(N)$.

An interesting peculiar case of inverse source problem is the case where g takes the form $g(x,t) = f(x)r(x,t)$ where r is a given function of $\mathcal{C}^1(\overline{\Omega} \times [0, T])$ and where $f \in L^2(\Omega)$ is the unknown function:

Theorem 4.2. *Let $r \in \mathcal{C}^1(\overline{\Omega} \times [0, T])$ be given such that $r(x,t) > 0$ for all $(x,t) \in \overline{\Omega} \times [0, T]$. Let $T > 0$, $t_0 \in (0, T)$ be given and consider $T_0 = (t_0 + T)/2$ and ω some non-empty open set of Ω. Assume that $\mu \leqslant \mu^\star(N)$ and $y_0 \in L^2(\Omega)$. Then, there exists $C = C(\Omega, \omega, t_0, T, r) > 0$, independent of $\mu \leqslant \mu^\star(N)$, such that for all $g_1 = f_1 r$ and $g_2 = f_2 r$ with $f_1, f_2 \in L^2(\Omega)$,*

$$\|f_1 - f_2\|^2_{L^2(\Omega_T)} \leqslant C\big\|y_1(\cdot, T_0) - y_2(\cdot, T_0)\big\|^2_{D(A_\mu)} + C\|y_{1,t} - y_{2,t}\|^2_{L^2(\omega_{t_0,T})}, \tag{9}$$

where y_1 and y_2 are the solutions of (2) *respectively associated to $g = f_1 r$ and $g = f_2 r$.*

As an application of Theorem 4.2, we deduce the following uniqueness result: assume that y_1 and y_2 are the solutions of (2) respectively associated to $g = f_1 r$ and $g = f_2 r$, then: if $A_\mu y_1(\cdot, T_0) = A_\mu y_2(\cdot, T_0)$ in Ω and $y_{1,t} = y_{2,t}$ in $\omega_{t_0,T}$ then $f_1 \equiv f_2$ in Ω.

5. Conclusion

In the case of the classical heat equation, this kind of problem with single measurement has largely been studied, see e.g. [10,11] and the references therein. In particular, using local Carleman estimates allows to obtain conditional Hölder estimates. Then global Carleman estimates have been used in [9] to provides stronger results of unconditional Lipschitz stability. Our result extends the result in [9] to the case of the heat equation with a singular potential in the case of a locally distributed observation of the solution. It could also be extended to the case of a potential with isolated multi-polar singularities (see the last comment in [7]). Let us also mention some recent analogous Lipschitz stability results for other non-classical heat equations: see e.g. [2] for the case of a discontinuous diffusion coefficient and [14,6] and next [4,5] for the case of a degenerate diffusion, respectively in 1-d and in higher dimensions.

References

[1] P. Baras, J. Goldstein, Remarks on the inverse square potential in quantum mechanics, in: Differential Equations, Birmingham, Ala., 1983, in: North-Holland Math. Stud., vol. 92, North-Holland, Amsterdam, 1984, pp. 31–35.

[2] A. Benabdallah, P. Gaitan, J. Le Rousseau, Stability of discontinuous diffusion coefficients and initial conditions in an inverse problem for the heat equation, SIAM J. Control Optim. 46 (5) (2007) 1849–1881.

[3] P. Cannarsa, P. Martinez, J. Vancostenoble, Carleman estimates for a class of degenerate parabolic operators, SIAM J. Control Optim. 47 (1) (2008) 1–19.

[4] P. Cannarsa, P. Martinez, J. Vancostenoble, Carleman estimates and null controllability for boundary-degenerate parabolic operators, C. R. Acad. Sci. Sér. I Math. 347 (3–4) (2009) 147–152.

[5] P. Cannarsa, P. Martinez, J. Vancostenoble, Carleman estimates for degenerate parabolic operators with applications, in press.

[6] P. Cannarsa, J. Tort, M. Yamamoto, Determination of source terms in a degenerate parabolic equation, in press.

[7] S. Ervedoza, Control and stabilization properties for a singular heat equation with an inverse-square potential, Comm. in PDE 33 11 (2008) 1996–2019.

[8] A.V. Fursikov, O.Yu. Imanuvilov, Controllability of Evolution Equations, Lecture Notes Series, vol. 34, Seoul National University, Seoul, Korea, 1996.

[9] O.Y. Imanuvilov, M. Yamamoto, Lipschitz stability in inverse parabolic problems by the Carleman estimate, Inverse Problems 14 (5) (1998) 1229–1245.

[10] V. Isakov, Inverse Source Problems, Mathematical Surveys and Monographs, vol. 34, American Mathematical Society, Providence, RI, 1990.

[11] M.V. Klibanov, A. Timonov, Carleman Estimates for Coefficient Inverse Problems and Numerical Applications, Inverse and Ill-posed Problems Series, VSP, Utrecht, 2004, iv+282 pp.

[12] P. Martinez, J. Vancostenoble, Carleman estimates for one-dimensional degenerate heat equations, J. Evol. Equ. 6 (2) (2006) 325–362.

[13] J.-P. Puel, M. Yamamoto, On a global estimate in a linear inverse hyperbolic problem, Inverse Problems 12 (6) (1996) 995–1002.

[14] J. Tort, Determination of source terms in a degenerate parabolic equation from a locally distributed observation, C. R. Acad. Sci. Sér., in press.

[15] J. Vancostenoble, E. Zuazua, Null controllability for the heat equation with singular inverse-square potentials, J. Funct. Anal. 254 (7) (2008) 1864–1902.

[16] J. Vancostenoble, E. Zuazua, Hardy inequalities, observability and control for the wave and Schrödinger equations with singular potentials, SIAM J. Math. Anal. 41 (4) (2009) 1508–1532.

[17] J.L. Vázquez, E. Zuazua, The Hardy inequality and the asymptotic behaviour of the heat equation with an inverse-square potential, J. Funct. Anal. 173 (1) (2000) 103–153.