C. R. Acad. Sci. Paris, Ser. I 348 (2010) 495–497

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Group Theory/Geometry

Complete reducibility and separable field extensions

Complète réductibilité et extensions de corps séparables

Michael Bate [a], Benjamin Martin [b], Gerhard Röhrle [c]

[a] *Department of Mathematics, University of York, York YO10 5DD, United Kingdom*
[b] *Mathematics and Statistics Department, University of Canterbury, Private Bag 4800, Christchurch 8140, New Zealand*
[c] *Fakultät für Mathematik, Ruhr-Universität Bochum, 44780 Bochum, Germany*

ARTICLE INFO

Article history:
Received 27 February 2010
Accepted after revision 6 April 2010
Available online 24 April 2010

Presented by Jean-Pierre Serre

ABSTRACT

Let G be a connected reductive linear algebraic group. The aim of this note is to settle a question of J-P. Serre concerning the behaviour of his notion of G-complete reducibility under separable field extensions. Part of our proof relies on the recently established Tits Centre Conjecture for the spherical building of the reductive group G.

© 2010 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

RÉSUMÉ

Soit G un groupe algébrique linéaire réductif connexe. Le but de cette Note est de répondre à une question de J-P. Serre concernant le comportement par extensions de corps séparables, de la notion de G-réductibilité complète qu'il a introduite. Une partie de nos arguments repose sur la solution récente de la conjecture du centre de Tits pour les immeubles sphériques du groupe réductif G.

© 2010 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

1. Introduction

Throughout, G denotes a connected reductive linear algebraic group defined over a field k. Following Serre, [13], a subgroup H of G is called *G-completely reducible over k* (*G-cr over k*) if whenever H is contained in a k-defined parabolic subgroup P of G, there exists a k-defined Levi subgroup of P containing H. In case V is a finite dimensional k-vector space and $G = \mathrm{GL}(V)$, a subgroup H of G is G-completely reducible over k precisely when V is a semisimple H-module, [13, 1.3, 3.2.2]. In this sense, Serre's notion generalizes the usual concept of complete reducibility in representation theory. For more details and further results on this notion, see [12,13,1,3], and [4].

The following theorem answers a question of Serre:

Theorem 1.1. *Suppose k_1/k is a separable extension of fields. Let G be a reductive group defined over k, and let H be a k-defined subgroup of G. Then H is G-completely reducible over k if and only if H is G-completely reducible over k_1.*

The reverse implication in Theorem 1.1 is proved in [4, Thm. 5.11]. The proof of [4, Thm. 5.11] rests on a general rationality result, [4, Thm. 3.1], concerning G-orbits in an affine variety. We present a proof of the forward direction of the statement in Section 3 based on the recently established Tits Centre Conjecture, Theorem 2.3.

E-mail addresses: meb505@york.ac.uk (M. Bate), B.Martin@math.canterbury.ac.nz (B. Martin), gerhard.roehrle@rub.de (G. Röhrle).

1631-073X/$ – see front matter © 2010 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.
doi:10.1016/j.crma.2010.04.013

Remarks 1.2. (i) In [4, Ex. 5.12], we showed that Theorem 1.1 holds when $G = \mathrm{GL}(V)$.

(ii) Theorem 1.1 was proved in [1, Thm. 5.8] for k perfect, by passing back and forth between k and its algebraic closure $\bar{k}$ and between k_1 and $\bar{k}$. In general this approach fails, because the extension $\bar{k}/k$ need not be separable.

(iii) There are examples showing that each implication in Theorem 1.1 fails without the separability assumption on the extension k_1/k; see [1, Ex. 5.11] and [3, Ex. 7.22].

2. The Centre Conjecture for spherical buildings

Let Δ_k denote the spherical building of G over k, [15, Sec. 5]: the simplices of Δ_k correspond to k-defined parabolic subgroups of G. Given a k-defined parabolic subgroup P of G, we denote the simplex corresponding to P in Δ_k by σ_P. Throughout, we identify Δ_k with its geometric realization, which is a bouquet of spheres [13].

An *apartment* in Δ_k consists of the simplices σ_P corresponding to all k-defined parabolic subgroups P of G that contain a fixed maximal k-split torus of G; it is a subcomplex whose geometric realization is a sphere. Any two points of Δ_k lie in a common apartment. We say that $x, y \in \Delta_k$ are *opposite* if they are opposite in some apartment that contains them both. It can be shown that if x and y are opposite in some apartment that contains them both, then they are opposite in any apartment that contains them both. If $x, y \in \Delta_k$ are not opposite, then there is a unique geodesic joining them, [13, §2.1.4]. Two simplices σ_P and σ_Q are said to be *opposite* if every point of σ_P is opposite a point of σ_Q, [13, §2.1.4]. In terms of parabolic subgroups of G, the simplices σ_P and σ_Q corresponding to k-defined parabolic subgroups P and Q of G are opposite in Δ_k if and only if $P \cap Q$ is a common Levi subgroup of P and Q (this Levi subgroup is then automatically k-defined).

Let Σ be a subcomplex of Δ_k. We say that Σ is *convex* if whenever $x, y \in \Sigma$ are not opposite, then Σ contains the geodesic between x and y, [13, §2.1].

Suppose Σ is a convex subcomplex of Δ_k. Serre has shown that Σ is *contractible* — that is, Σ has the homotopy type of a point — if and only if there exists a point of Σ which has no opposite in Σ; see [13, §2.2]. The following terminology is due to Serre [13, Def. 2.2.1]:

Definition 2.1. Let Σ be a convex subcomplex of Δ_k. We say that Σ is *Δ_k-completely reducible* (or Δ_k-cr) if every simplex in Σ has an opposite in Σ.

Serre has shown that the group-theoretic definition of G-complete reducibility over k has the following building-theoretic interpretation, [13]: Given a subgroup H of G, let

$$\Delta_k^H = \{\sigma_P \mid P \text{ is a } k\text{-defined parabolic subgroup containing } H\}.$$

Then Δ_k^H is a convex subcomplex of Δ_k [13, Prop. 3.1], the *fixed point subcomplex* of Δ_k under the action of H, and H is G-completely reducible over k if and only if Δ_k^H is Δ_k-cr, [13, 2.3.1, 3.2]. Equivalently, H is *not* G-completely reducible over k if and only if Δ_k^H is contractible.

Definition 2.2. Let Σ be a subcomplex of Δ_k and let $x \in \Sigma$. Let Γ be a group which acts on Δ_k by means of building automorphisms, [15], i.e., suppose there is a homomorphism $\Gamma \to \mathrm{Aut}\,\Delta_k$, where $\mathrm{Aut}\,\Delta_k$ is the group of building automorphisms of Δ_k. We say that x is a *Γ-centre of Σ* if x is fixed by any element of Γ that stabilizes Σ setwise.

The following theorem is known as the "Centre Conjecture" of J. Tits, cf. [14, Lem. 1.2], [12, §4], [13, §2.4], [16], [9, Ch. 2, §3], [11, Conj. 3.3]. It has recently been proved in a series of intricate case-by-case arguments by B. Mühlherr and J. Tits [8] (G of classical type or type G_2), B. Leeb and C. Ramos-Cuevas [7] (G of type F_4 or E_6) and C. Ramos-Cuevas [10] (G of type E_7 or E_8).

Theorem 2.3 *(Tits' Centre Conjecture). Let Σ be a convex contractible subcomplex of Δ_k. Then Σ has an* $\mathrm{Aut}\,\Delta_k$*-centre.*

Remark 2.4. Suppose G is semisimple and k is a perfect field. It follows from [15, 5.7.2] that $\mathrm{Aut}\,G$ is an algebraic group also defined over k. In [4, Thm. 5.31], we give a uniform proof of the following special case of the Centre Conjecture: Let H be a subgroup of G. If Δ_k^H is contractible, i.e., if Δ_k^H is not Δ_k-cr, then Δ_k^H admits an $(\mathrm{Aut}\,G)(k)$-centre. The proof of this result in [4] utilizes methods from geometric invariant theory and the concept of optimal destabilizing parabolic subgroups.

Let k be a field, let k_s denote its separable closure, and let $\bar{k}$ denote its algebraic closure. Note that $k_s = \bar{k}$ if k is perfect. Thanks to [15, 5.7.2], $\Gamma := \mathrm{Gal}(k_s/k)$ acts on $\Delta_{\bar{k}}$ via building automorphisms. In [4, Thm. 5.33], we show that if H is a k-defined subgroup of G such that $\Delta_{\bar{k}}^H$ is contractible, then $\Delta_{\bar{k}}^H$ admits a Γ-centre. The proof of [4, Thm. 5.33] rests on a rationality result concerning G-cr subgroups of G, [4, Prop. 5.14(iii)].

Both [4, Thm. 5.31] and [4, Thm. 5.33] improve on [2, Thm. 3.1].

3. Proof of Theorem 1.1

As noted above the reverse implication of Theorem 1.1 is proved in [4, Thm. 5.11]. We deduce the other direction with the aid of Theorem 2.3.

Suppose k_1/k is an algebraic separable extension of fields and let Δ_k and Δ_{k_1} denote the buildings of G over k and k_1, respectively. By the reverse implication, one may suppose that k_1/k is Galois. Then the Galois group $\Gamma := \mathrm{Gal}(k_1/k)$ acts simplicially on Δ_{k_1}, i.e., Γ permutes the set of k_1-defined parabolic subgroups of G. Moreover, the subcomplex of Δ_{k_1} consisting of Γ-stable simplices is just Δ_k.

It is convenient to reduce to the case when H is not contained in any k-defined Levi subgroup of any proper k-defined parabolic subgroup of G. To do this, we let L be minimal such that L is a k-defined Levi subgroup of some k-defined parabolic subgroup P of G and $H \subseteq L$. Then L is also k_1-defined, and by a result of Serre, [13, Prop. 3.2], H is G-completely reducible over k (resp. k_1) if and only if H is L-completely reducible over k (resp. k_1). Now if L' is a k-defined Levi subgroup of some proper k-defined parabolic subgroup Q of L, then $QR_u(P)$ is a k-defined parabolic subgroup of G, and L' is a Levi subgroup of $QR_u(P)$, [6, Prop. 4.4]. Since L is minimal among those k-defined Levi subgroups of k-defined parabolic subgroups of G that contain H, H cannot be contained in L'. By replacing G with L, we can now assume that H is not contained in any k-defined Levi subgroup of any proper k-defined parabolic subgroup of G.

Suppose that H is not G-completely reducible over k_1. Then $\Delta_{k_1}^H$ is contractible, and since H is k-defined, $\Delta_{k_1}^H$ is Γ-stable. Since $\Delta_{k_1}^H$ is a convex contractible subcomplex of Δ_{k_1}, it follows from Theorem 2.3 that Γ fixes a point of $\Delta_{k_1}^H$, and this point lies in some (minimal) simplex σ_P, where P is a proper k_1-defined parabolic subgroup of G. Since the action of Γ on Δ_{k_1} is simplicial, P is stabilized by Γ, which is equivalent to saying that P is k-defined. Now, by assumption, H is not contained in any k-defined Levi subgroup of P, so H is not G-completely reducible over k. This completes the proof of Theorem 1.1.

Remark 3.1. In [5, Thm. 4.13], we prove a generalization of the reverse implication of Theorem 1.1 in the setting of "relative complete reducibility". The arguments above used to derive the forward direction of Theorem 1.1 do not apply to this more general situation, as the relevant subset in Δ_{k_1} is only a convex subset but not a subcomplex of Δ_{k_1}. Thus Theorem 2.3 does not apply.

Acknowledgements

The paper was written during a stay of the first and third authors at the Max Planck Institute for Mathematics in Bonn. The authors also acknowledge financial support from the DFG-priority program SPP1388 "Representation Theory", Marsden Grant UOC0501, and The Royal Society.

References

[1] M. Bate, B. Martin, G. Röhrle, A geometric approach to complete reducibility, Invent. Math. 161 (1) (2005) 177–218.
[2] M. Bate, B. Martin, G. Röhrle, On Tits' centre conjecture for fixed point subcomplexes, C. R. Math. Acad. Sci. Paris, Ser. I 347 (2009) 353–356.
[3] M. Bate, B. Martin, G. Röhrle, R. Tange, Complete reducibility and separability, Trans. AMS 362 (8) (2010) 4283–4311.
[4] M. Bate, B. Martin, G. Röhrle, R. Tange, Closed orbits and uniform S-instability in geometric invariant theory, preprint, arXiv:0904.4853v3 [math.AG], 2009.
[5] M. Bate, B. Martin, G. Röhrle, R. Tange, Complete reducibility and conjugacy classes of tuples in algebraic groups and Lie algebras, preprint, arXiv:0905.0065v2 [math.GR], 2009.
[6] A. Borel, J. Tits, Groupes réductifs, Inst. Hautes Études Sci. Publ. Math. 27 (1965) 55–150.
[7] B. Leeb, C. Ramos-Cuevas, The center conjecture for spherical buildings of types F_4 and E_6, preprint, arXiv:0905.0839v1.
[8] B. Mühlherr, J. Tits, The Centre Conjecture for non-exceptional buildings, J. Algebra 300 (2) (2006) 687–706.
[9] D. Mumford, J. Fogarty, F. Kirwan, Geometric Invariant Theory, third edition, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 34, Springer-Verlag, Berlin, 1994.
[10] C. Ramos-Cuevas, The center conjecture for thick spherical buildings, preprint, arXiv:0909.2761v1.
[11] G. Rousseau, Immeubles sphériques et théorie des invariants, C.R.A.S. 286 (1978) 247–250.
[12] J-P. Serre, La notion de complète réductibilité dans les immeubles sphériques et les groupes réductifs, Séminaire au Collège de France, résumé dans [17, pp. 93–98], 1997.
[13] J-P. Serre, Complète réductibilité, Séminaire Bourbaki, 56ème année (2003–2004) no. 932.
[14] J. Tits, Groupes semi-simples isotropes, Colloq. Théorie des Groupes Algébriques, Bruxelles, 1962.
[15] J. Tits, Buildings of Spherical Type and Finite BN-Pairs, Lecture Notes in Math., vol. 386, Springer-Verlag, 1974.
[16] J. Tits, Quelques cas d'existence d'un centre pour des ensembles de chambres qui sont convexes, non vides et ne contiennent pas de paires de chambres opposées, Séminaire au Collège de France, résumé dans [17, pp. 98–101], 1997.
[17] J. Tits, Théorie des groupes, Résumé des Cours et Travaux, Annuaire du Collège de France, 97e année (1996–1997) 89–102.