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Mathematical Analysis

Asymptotic behavior of polynomially bounded operators

Comportement asymptotique des opérateurs polynomialement bornés

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ABSTRACT

Let T be a polynomially bounded operator on a complex Banach space and let A_T be the smallest uniformly closed (Banach) algebra that contains T and the identity operator. It is shown that for every $S \in A_T$,

$$\lim_{n\to\infty} \|T^n S\| = \sup_{\xi\in\sigma_u(T)} |\widehat{S}(\xi)|,$$

where $\widehat{S}$ is the Gelfand transform of S and $\sigma_u(T) := \sigma(T) \cap \Gamma$ is the unitary spectrum of T; $\Gamma = \{z \in \mathbb{C}\colon |z| = 1\}$.

RÉSUMÉ

Soit T un opérateur polynomialement borné sur un espace de Banach et soit A_T la plus petite algèbre de Banach uniformement fermé contenant T et l'identité. Il est montré dans cet article que pour tout $S \in A_T$,

$$\lim_{n\to\infty} \|T^n S\| = \sup_{\xi\in\sigma_u(T)} |\widehat{S}(\xi)|,$$

où $\widehat{S}$ est la transformée de Gelfand et $\sigma_u(T) := \sigma(T) \cap \Gamma$ est la spectre unitaire de T ; $\Gamma := \{z \in \mathbb{C}\colon |z| = 1\}$.

1. Introduction and preliminaries

Let $B(X)$ be the algebra of all bounded, linear operators on the complex Banach space X. For $T \in B(X)$, we denote by $\sigma(T)$ the spectrum and by $R(z,T) = (z-T)^{-1}$ the resolvent of T. We have written $D = \{z \in \mathbb{C}\colon |z| < 1\}$ and $\Gamma = \{z \in \mathbb{C}\colon |z| = 1\}$. By $A(D)$ we will denote the disc-algebra and by $H^\infty := H^\infty(D)$, the algebra of all bounded analytic functions on D.

Recall that the set $\sigma_u(T) := \sigma(T) \cap \Gamma$ is called *unitary spectrum* of $T \in B(X)$. If $T \in B(X)$, we let A_T denote the closure in the uniform operator topology of all polynomials in T. Then, A_T is a commutative unital Banach algebra. The maximal ideal space of A_T can be identified with $\sigma_{A_T}(T)$, the spectrum of T with respect to the algebra A_T [4, Theorem 4.5.1]. By $\widehat{S}$ we will denote the Gelfand transform of $S \in A_T$. Since $\sigma(T) \subset \sigma_{A_T}(T)$, for every $\xi \in \sigma(T)$ there exists a multiplicative functional ϕ_ξ on A_T such that $\phi_\xi(T) = \xi$. Here, and in the sequel, instead of $\widehat{S}(\phi_\xi)$ $(= \phi_\xi(S))$, $\xi \in \sigma(T)$, we will use the notation $\widehat{S}(\xi)$. Note that $\xi \mapsto \widehat{S}(\xi)$ is a continuous function on $\sigma(T)$.

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Recall that an operator $T \in B(X)$ is said to be *polynomially bounded* if for all polynomials P we have

$$\|P(T)\| \leqslant \|P\|_\infty.$$

The von Neumann inequality asserts that every Hilbert space contraction is polynomially bounded. This result does not extend to Banach space contractions. We see that every polynomially bounded operator is a contraction.

Note that every polynomially bounded operator $T \in B(X)$ admits an $A(D)$-functional calculus. This means that there exists a contractive algebra-homomorphism $h : A(D) \mapsto A_T$ (with dense range) such that $1 \mapsto I$ and $z \mapsto T$. We will use the notation $f(T) := h(f),\ f \in A(D)$. Thus we have $\|f(T)\| \leqslant \|f\|_\infty$ for all $f \in A(D)$. We also have

$$\|f(T)\| \geqslant \sup_{\xi \in \sigma(T)} |f(\xi)|, \quad f \in A(D). \tag{1.1}$$

The Esterle–Strouse–Zouakia Theorem [1] asserts that if T is a contraction on a Hilbert space and $f \in A(D)$ vanishes on $\sigma_u(T)$, then $\lim_{n\to\infty} \|T^n f(T)\| = 0$. The similar result holds for polynomially bounded operators [3] (for related results see also [2,5,8,10]). We see that under the assumptions of Esterle–Strouse–Zouakia Theorem the Lebesgue measure of $\sigma_u(T)$ is necessarily zero. In this note, we address the problem whether quantitative versions of the above results hold.

2. The main result

The main result of this Note is the following theorem:

Theorem 2.1. *If $T \in B(X)$ is a polynomially bounded operator, then for every $S \in A_T$,*

$$\lim_{n\to\infty} \|T^n S\| = \sup_{\xi \in \sigma_u(T)} |\widehat{S}(\xi)|.$$

For the proof we need some preliminary results. Suppose that $V \in B(X)$ is an invertible isometry. By $A_{V,V^{-1}}$ we will denote the closure in the uniform operator topology of all trigonometric polynomials in V. Then, $A_{V,V^{-1}}$ is a commutative unital Banach algebra. If V is polynomially bounded, then V admits $C(\Gamma)$-functional calculus (for more details see [3]), that is, there exists a contractive algebra-homomorphism $h : C(\Gamma) \mapsto A_{V,V^{-1}}$ (with dense range) such that $1 \mapsto I$, $e^{it} \mapsto V$ and $e^{-it} \mapsto V^{-1}$. We will use the notation $f(T) := h(f),\ f \in C(\Gamma)$. Thus we have $\|f(V)\| \leqslant \|f\|_\infty$ for all $f \in C(\Gamma)$. We also have

$$\|f(V)\| \geqslant \sup_{\xi \in \sigma(V)} |f(\xi)|, \quad f \in C(\Gamma). \tag{2.1}$$

Proposition 2.2. *If V is a polynomially bounded isometry, then the following assertions hold*:

a) *If V is invertible, then the algebra $A_{V,V^{-1}}$ (in the case when $\sigma(V) \neq \Gamma$, then the algebra A_V) is isometric and algebra isomorphic to $C(\sigma(V))$;*
b) *For every $f \in A(D)$, $\|f(V)\| = \sup_{\xi \in \sigma_u(V)} |f(\xi)|$.*

Proof. a) For a given $f \in C(\Gamma)$ and $\varepsilon > 0$, there exists a function $g \in C(\Gamma)$ such that $f(\xi) = g(\xi)$ on $\sigma(V)$ and $\|g\|_\infty \leqslant \sup_{\xi\in\sigma(V)} |f(\xi)| + \varepsilon$. Since $f(V) = g(V)$, we have

$$\|f(V)\| = \|g(V)\| \leqslant \|g\|_\infty \leqslant \sup_{\xi \in \sigma(V)} |f(\xi)| + \varepsilon.$$

Since ε was arbitrary, we obtain $\|f(V)\| \leqslant \sup_{\xi\in\sigma(V)} |f(\xi)|$. The opposite inequality follows from (2.1). Note also that if $\sigma(V) \neq \Gamma$, then $V^{-1} \in A_V$.

b) It is well known that if V is a non-invertible isometry, then $\sigma(V) = \overline{D}$. Now, the assertion follows from a) and (1.1). □

The following result is well known (see, for instance [3] and [7, Lemma 2.1]):

Lemma 2.3. *If $T \in B(X)$ is a contraction, then there exists a Banach space Y, a linear contractive operator $J : X \mapsto Y$ with dense range and an isometry V on Y such that*:
i) $VJ = JT$; ii) $\|Jx\| = \lim_{n\to\infty} \|T^n x\|$ *for all* $x \in X$; iii) $\sigma(V) \subset \sigma(T)$.

The triple (Y, J, V) will be called the *limit isometry* associated to T. It is easy to verify that if $T \in B(X)$ is polynomially bounded, then the limit isometry V associated to T is also polynomially bounded (see also [3]).

For a given $T \in B(X)$ and $x \in X$, we define $\rho_T(x)$ to be the set of all $\lambda \in \mathbb{C}$ for which there exists a neighborhood U_λ of λ with $u(z)$ analytic on U_λ having values in X such that $(zI - T)u(z) = x$ on U_λ. This set is open and contains the resolvent set $\rho(T)$ of T. By definition, the *local spectrum* of T at x, denoted by $\sigma_T(x)$, is the complement of $\rho_T(x)$, so it is a closed subset of $\sigma(T)$.

Let $T \in B(X)$ be a contraction and let (Y, J, V) be the limit isometry associated to T. We claim that $\sigma_V(Jx) \subset \sigma_T(x)$ for every $x \in X$. To see this, let $\lambda \in \rho_T(x)$. Then there exists a neighborhood U_λ of λ with X-valued function $u(z)$ analytic on U_λ such that $(zI - T)u(z) = x$, $z \in U_\lambda$. It follows that $(zJ - JT)u(z) = Jx$. In view of Lemma 2.3i), since $JT = VJ$, we get $(zI - V)Ju(z) = Jx$, $z \in U_\lambda$. This shows that $\lambda \in \rho_V(Jx)$.

The following lemma was proved in [3, Lemma 1.3]:

Lemma 2.4. *If $V \in B(X)$ is an isometry and $x \in X$ is a cyclic vector of V, then*

$$\sigma_u(V) = \sigma_V(x) \cap \Gamma.$$

Proposition 2.5. *If $T \in B(X)$ is a polynomially bounded operator, then for every $f \in A(D)$ and $x \in X$,*

$$\lim_{n\to\infty} \left\| T^n f(T)x \right\| \leqslant \sup_{\xi \in \sigma_T(x)\cap\Gamma} \left| f(\xi) \right| \|x\|.$$

Proof. For a given $x \in X$, let E be the closed linear span of the set $\{T^n x\colon\ n \geqslant 0\}$. Then, E is a T-invariant subspace of X. Clearly, the restriction $T|_E$ of T to E is also a polynomially bounded operator. Let (Y, J, V) be the limit isometry associated to $T|_E$. As we already noted above that $\sigma_V(Jx) \subset \sigma_{T|_E}(x)$ and therefore, $\sigma_V(Jx) \cap \Gamma \subset \sigma_{T|_E}(x) \cap \Gamma$.

Let us show that $\sigma_{T|_E}(x) \cap \Gamma \subset \sigma_T(x) \cap \Gamma$. Let $\xi \in \rho_T(x) \cap \Gamma$ and let $\pi : X \mapsto X/E$ be the canonical mapping. Then there exists a neighborhood U_ξ of ξ with $u(z)$ analytic on U_ξ having values in X such that $(zI - T)u(z) = x$ on U_ξ. Since

$$u(z) = R(z, T)x = \sum_{n=0}^{\infty} z^{-n-1} T^n x \in E,$$

for all $z \in U_\xi$ with $|z| > 1$, we have $\pi u(z) = 0$ for all $z \in U_\xi$ with $|z| > 1$. By uniqueness theorem, $\pi u(z) = 0$ for all $z \in U_\xi$, so that $u(z) \in E$. Hence, we have $(zI - T|_E)u(z) = x$ on U_ξ. This shows that $\xi \in \rho_{T|_E}(x) \cap \Gamma$.

Consequently, we have

$$\sigma_V(Jx) \cap \Gamma \subset \sigma_T(x) \cap \Gamma. \tag{2.2}$$

From Lemma 2.3i) we can deduce that Jx is a cyclic vector of V. In view of Lemma 2.4 and (2.2), we obtain that

$$\sigma_u(V) = \sigma_V(Jx) \cap \Gamma \subset \sigma_T(x) \cap \Gamma. \tag{2.3}$$

Now, let $f \in A(D)$ be given. By Lemma 2.3i), we can write

$$f(V)J = Jf(T|_E) = J\big(f(T)|_E\big),$$

so that $f(V)Jx = Jf(T)x$. Since V is polynomially bounded, combining Lemma 2.3ii), Proposition 2.2 and (2.3), we have

$$\lim_{n\to\infty} \left\| T^n f(T)x \right\| = \left\| Jf(T)x \right\| = \left\| f(V)Jx \right\| \leqslant \sup_{\xi\in\sigma_u(V)} \left| f(\xi) \right| \|Jx\| \leqslant \sup_{\xi\in\sigma_T(x)\cap\Gamma} \left| f(\xi) \right| \|x\|. \qquad \square$$

Proof of Theorem 2.1. Let $S \in A_T$. For every $\xi \in \sigma_u(T)$, there exists a multiplicative functional ϕ_ξ on A_T such that $\phi_\xi(T) = \xi$. Since ϕ_ξ has norm one, we have $\|T^n S\| \geqslant |\phi_\xi(T^n S)| = |\xi^n \widehat{S}(\xi)| = |\widehat{S}(\xi)|$. It follows that

$$\lim_{n\to\infty} \left\| T^n S \right\| \geqslant \sup_{\xi\in\sigma_u(T)} \left| \widehat{S}(\xi) \right|.$$

To prove the opposite inequality, let L_T be the left multiplication operator on $B(X)$; $L_T Q = TQ$, $Q \in B(X)$. Clearly, L_T is a polynomially bounded operator. In view of Proposition 2.5, we can write

$$\lim_{n\to\infty} \left\| T^n f(T)Q \right\| \leqslant \sup_{\xi\in\sigma_{L_T}(Q)\cap\Gamma} \left| f(\xi) \right| \|Q\|,$$

for all $Q \in B(X)$ and $f \in A(D)$. It is easy to verify that $\sigma_{L_T}(I) \cap \Gamma \subset \sigma_u(T)$. Now, by putting in the last inequality $Q = I$, we obtain

$$\lim_{n\to\infty} \left\| T^n f(T) \right\| \leqslant \sup_{\xi\in\sigma_u(T)} \left| f(\xi) \right|, \quad f \in A(D).$$

For a given $\varepsilon > 0$, there exists a function $f \in A(D)$ such that $\|S - f(T)\| \leqslant \varepsilon$. It follows that $\|T^n S\| \leqslant \|T^n f(T)\| + \varepsilon$ $(n \in \mathbb{N})$, and

$$\sup_{\xi\in\sigma_u(T)}\left|f(\xi)\right| \leqslant \sup_{\xi\in\sigma_u(T)}\left|\widehat{S}(\xi)\right| + \varepsilon.$$

Hence, we can write

$$\lim_{n\to\infty}\left\|T^n S\right\| \leqslant \lim_{n\to\infty}\left\|T^n f(T)\right\| + \varepsilon \leqslant \sup_{\xi\in\sigma_u(T)}\left|f(\xi)\right| + \varepsilon \leqslant \sup_{\xi\in\sigma_u(T)}\left|\widehat{S}(\xi)\right| + 2\varepsilon.$$

Since ε was arbitrary, we have

$$\lim_{n\to\infty}\left\|T^n S\right\| \leqslant \sup_{\xi\in\sigma_u(T)}\left|\widehat{S}(\xi)\right|,$$

which finishes the proof. □

Corollary 2.6. *If T is a contraction on a Hilbert space, then for every $S \in A_T$,*

$$\lim_{n\to\infty}\left\|T^n S\right\| = \sup_{\xi\in\sigma_u(T)}\left|\widehat{S}(\xi)\right|.$$

3. Applications

Let T be a contraction on a Hilbert space H such that $\lim_{n\to\infty}\|T^n x\| = 0$ and $\lim_{n\to\infty}\|T^{*n}x\| = 0$ for every $x \in H$. Moreover, assume that $\dim(I - TT^*)H = \dim(I - T^*T)H = 1$. According to the well-known Model Theorem of Nagy–Foias [9], T is unitarily equivalent to its model operator $M_\varphi = P_\varphi S|_{K_\varphi}$ acting on the model space $K_\varphi := H^2 \ominus \varphi H^2$, where φ is an inner function, $Sf = zf$ is the shift operator on the Hardy space H^2 and P_φ is the orthogonal projection from H^2 onto K_φ. It follows that for every $f \in H^\infty$, the operator $f(T)$ is unitarily equivalent to $f(M_\varphi) = P_\varphi f(S)|_{K_\varphi}$. As is known [6, p. 235], $\|f(M_\varphi)\| = \operatorname{dist}(f, \varphi H^\infty)$. Hence, we have $\|T^n f(T)\| = \operatorname{dist}(z^n f, \varphi H^\infty) = \operatorname{dist}(f, \bar{z}^n \varphi H^\infty)$. The *unitary spectrum* $\Sigma_u(\varphi)$ of φ is defined as

$$\Sigma_u(\varphi) = \left\{\xi \in \Gamma\colon \liminf_{z\in D,\ z\to\xi}\left|\varphi(z)\right| = 0\right\}.$$

It follows from the Lipschitz–Moeller Theorem [6, p. 81] that $\sigma_u(T) = \Sigma_u(\varphi)$. Now, applying Theorem 2.1, we have the following.

Corollary 3.1. *If φ is an inner function, then for every $f \in A(D)$,*

$$\lim_{n\to\infty}\operatorname{dist}\left(f, \bar{z}^n\varphi H^\infty\right) = \sup_{\xi\in\Sigma_u(\varphi)}\left|f(\xi)\right|.$$

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