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Partial Differential Equations/Mathematical Problems in Mechanics

Existence and uniqueness of solutions for the hydrostatic Euler equations on a bounded domain with analytic data

Existence locale et unicité de solutions de l'équation d'Euler hydrostatique dans un ouvert borné avec des données analytiques

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ARTICLE INFO

Article history:

Received 14 March 2010

Accepted 23 March 2010

Presented by Philippe G. Ciarlet

ABSTRACT

We address the question of well-posedness in spaces of analytic functions for the hydrostatic incompressible Euler equations (inviscid primitive equations) on domains with boundary, with a novel side-boundary condition.

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R É S U M É

On étudie le caractère bien posé dans des espaces de fonctions analytiques de l'équation d'Euler hydrostatique pour un fluide incompressible (équations primitives non-visqueuses) sur des domaines à bords, avec une nouvelle condition de bord.

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Version française abrégée

Plusieurs modèles ont été proposés pour l'étude des fluides géophysiques en équilibre hydrostatique cf. [8,9,12,19]. Dans cette Note nous considérons le modèle hydrostatique sans viscosité qui est classique dans la littérature géophysique, voir par exemple [12] ; voir aussi P.-L. Lions [8, Section 4.6], où la question d'existence et d'unicité des solutions a été posée. Ces équations sont obtenues formellement à partir des équations d'Euler des fluides incompressibles [4,8]. Le problème posé est de trouver un champ de vitesses $\mathbf{u} = (v_1, v_2, w) = (\mathbf{v}, w)$, une pression p , et une densité ρ qui vérifient

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} + w \partial_z \mathbf{v} + \nabla(p - g\psi) + f \mathbf{v}^\perp = 0, \quad (1)$$

$$\partial_t \rho + (\mathbf{v} \cdot \nabla) \rho + w \partial_z \rho = 0, \quad \operatorname{div} \mathbf{v} + \partial_z w = 0, \quad \partial_z p = 0, \quad (2)$$

$$\psi(x, z, t) = \int_0^z \rho(x, \zeta, t) d\zeta, \quad (3)$$

dans $\mathcal{D} \times (0, T)$, pour un $T > 0$. Ici $\mathcal{D} = \mathcal{M} \times (0, h) = \{(x_1, x_2, z) = (x, z) \in \mathbb{R}^3 : x \in \mathcal{M}, 0 < z < h\}$ est un cylindre de hauteur h , où $\mathcal{M} \subset \mathbb{R}^2$ est un domaine bidimensionnel de frontière analytique réelle. Pour les notations voir la version

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anglaise. Les conditions aux limites sur le haut et le bas du cylindre $\Gamma_z = \mathcal{M} \times \{0, h\}$ et sur la frontière latérale $\Gamma_x = \partial\mathcal{M} \times (0, h)$ sont

$$w(x, z, t) = 0, \quad \text{sur } \Gamma_z \times (0, T), \quad \text{et} \quad \int_0^h \mathbf{v}(x, z, t) \, dz \cdot \mathbf{n} = 0, \quad \text{sur } \Gamma_x \times (0, T), \quad (4)$$

où \mathbf{n} est le vecteur unitaire normal à $\partial\mathcal{M}$. On note que nous n'avons pas d'équation d'évolution pour w , qui est déterminé par (4) et la condition d'incompressibilité qui donnent par intégration $w(x, z, t) = -\int_0^z \operatorname{div} \mathbf{v}(x, \zeta, t) \, d\zeta$, et

$$\int_0^h \operatorname{div} \mathbf{v}(x, z, t) \, dz = 0, \quad (5)$$

pour tout $x \in \mathcal{M}$ et $0 < t < T$. Nous considérons des données initiales qui sont analytiques réelles $\mathbf{v}(x, z, 0) = \mathbf{v}_0(x, z)$, et $\rho(x, z, 0) = \rho_0(x, z)$ dans \mathcal{D} et qui satisfont les conditions (4) et (5).

Le problème de l'existence locale de solutions pour l'équation d'Euler hydrostatique était encore ouvert à notre connaissance (cf. [8]). Les difficultés principales sont liées aux conditions aux limites (cf. [16]), et au phénomène de perte de dérivée qui ne permet pas d'obtenir des estimations dans les espaces de Sobolev classiques (cf. [4,13]). Le seul résultat d'existence connu des auteurs a été obtenu dans le cas de la dimension deux d'espace par Brenier [3] sous une hypothèse de convexité en z et de périodicité dans la direction x . Notons par ailleurs les progrès obtenus dans le cas linéaire avec des conditions aux limites non-locales dans [14,15].

Dans cette Note, nous introduisons la condition aux limites (4), et nous montrons l'existence et l'unicité de solutions de (1)–(4) dans le cas de la dimension d'espace 2 (c-à-d \mathcal{M} et (\mathbf{u}, ρ, p) indépendants de x_2), et dans le cas de la dimension 3, avec \mathcal{M} un demi-plan ou un domaine périodique. La solution que nous construisons est analytique réelle jusqu'à la frontière et elle est unique dans cette classe.

Nous énonçons à présent les deux résultats principaux de cette Note pour les dimensions deux et trois et nous renvoyons le lecteur à la version anglaise pour les notations et les démonstrations.

Théorème 0.1. *Supposons que les fonctions \mathbf{u}, p, ρ sont indépendantes de x_2 , et soit $r \geq 2$. Supposons que $\mathbf{v}_0, \rho_0 \in X_{\tau_0}$, pour un $\tau_0 > 0$, et supposons que \mathbf{v}_0 satisfait les conditions (4) et (5). Alors, il existe $T_* = T_*(r, g, \tau_0, \|(\mathbf{v}_0, \rho_0)\|_{X_{\tau_0}}) > 0$, et une solution analytique réelle unique $(\mathbf{v}(\cdot, t), \rho(\cdot, t))$ du problème (1)–(4) avec un rayon d'analyticité $\tau(t)$, tels que*

$$\begin{aligned} & \|(\mathbf{v}(t), \rho(t))\|_{X_{\tau(t)}} + Cg \int_0^t e^{g(t-s)} \|(\mathbf{v}(s), \rho(s))\|_{Y_{\tau(s)}} \, ds \\ & + C \|(\mathbf{v}_0, \rho_0)\|_{X_{\tau_0}} e^{gt} \int_0^t (1 + \tau^{-2}(s)) \|(\mathbf{v}(s), \rho(s))\|_{Y_{\tau(s)}} \, ds \leq \|(\mathbf{v}_0, \rho_0)\|_{X_{\tau_0}} e^{gt}, \end{aligned} \quad (6)$$

pour tout $t \in [0, T_*)$, où $C = C(\mathcal{D})$ est une constante positive fixée. De plus, le rayon d'analyticité de la solution, $\tau : [0, T_*) \mapsto \mathbb{R}_+$, peut être calculé explicitement à partir de l'équation (29) ci-dessous.

Dans le cas de la dimension trois d'espace, la condition aux limites (4) nous permet de calculer la pression comme la solution d'un problème elliptique de type Neumann (cf. [18]). Nous avons alors

Théorème 0.2. *Nous supposons que $r \geq 5/2$ et que \mathcal{M} est le demi plan supérieur $\{x_1 > 0\}$ ou bien le domaine périodique $(0, 2\pi)^2$. Nous supposons aussi que $\mathbf{v}_0, \rho_0 \in X_{\tau_0}$, pour un $\tau_0 > 0$, et que \mathbf{v}_0 satisfait les conditions (4) et (5). Alors il existe $T_* = T_*(r, f, g, \tau_0, \|(\mathbf{v}_0, \rho_0)\|_{X_{\tau_0}}) > 0$, et une solution analytique réelle unique $(\mathbf{v}(\cdot, t), \rho(\cdot, t))$ du problème (1)–(4) avec un rayon d'analyticité $\tau(t)$, tels que*

$$\begin{aligned} & \|(\mathbf{v}(t), \rho(t))\|_{X_{\tau(t)}} + Cg \int_0^t e^{C_1(t-s)} \|(\mathbf{v}(s), \rho(s))\|_{Y_{\tau(s)}} \, ds \\ & + C \|(\mathbf{v}_0, \rho_0)\|_{X_{\tau_0}} e^{C_1 t} \int_0^t (1 + \tau^{-5/2}(s)) \|(\mathbf{v}(s), \rho(s))\|_{Y_{\tau(s)}} \, ds \leq \|(\mathbf{v}_0, \rho_0)\|_{X_{\tau_0}} e^{C_1 t}, \end{aligned} \quad (7)$$

pour tout $t \in [0, T_*)$, où $C = C(\mathcal{D})$ et $C_1 = C_1(f, g)$ sont des constantes positives fixes. De plus, le rayon d'analyticité de la solution, $\tau : [0, T_*) \mapsto \mathbb{R}_+$, peut être calculé explicitement.

1. Introduction

Several models have been proposed to study incompressible homogeneous geophysical flows in the hydrostatic limit; cf. [8,9,12,19] and the references therein. In this Note we consider the inviscid hydrostatic model which is classical in geophysical fluid mechanics; see for instance [12] and also P.-L. Lions [8, Section 4.6], where the author raises the question of existence and uniqueness of solutions. These equations are formally derived from the 3D incompressible Euler equations [4,8]. The problem is to find a velocity field $\mathbf{u} = (v_1, v_2, w) = (\mathbf{v}, w)$, a pressure scalar p , and a density scalar ρ solving

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} + w \partial_z \mathbf{v} + \nabla(p - g\psi) + f \mathbf{v}^\perp = 0, \tag{8}$$

$$\operatorname{div} \mathbf{v} + \partial_z w = 0, \tag{9}$$

$$\partial_z p = 0, \tag{10}$$

$$\psi(x, z, t) = \int_0^z \rho(x, \zeta, t) \, d\zeta, \tag{11}$$

$$\partial_t \rho + (\mathbf{v} \cdot \nabla) \rho + w \partial_z \rho = 0, \tag{12}$$

in $\mathcal{D} \times (0, T)$, for some $T > 0$. Here $\mathcal{D} = \mathcal{M} \times (0, h) = \{(x_1, x_2, z) = (x, z) \in \mathbb{R}^3: x \in \mathcal{M}, 0 < z < h\}$ is a cylinder of height h , where $\mathcal{M} \subset \mathbb{R}^2$ is a smooth domain with real-analytic boundary. Denote by $\operatorname{div}, \nabla$, and Δ the corresponding 2D operators acting on $x = (x_1, x_2)$, while $\partial_z = \partial/\partial z$. Also, let $\mathbf{v}^\perp = (v_2, -v_1)$ be the first two components of $\mathbf{u} \times e_3$, f is the strength of the rotation, and g is the gravitational constant. It follows from (10) and (11) that the full pressure $P(x, z, t) = p(x, t) - g\psi(x, z, t)$ satisfies $\partial_z P = -\rho g$. The boundary conditions at the top and bottom $\Gamma_z = \mathcal{M} \times \{0, h\}$ and at the side $\Gamma_x = \partial\mathcal{M} \times (0, h)$ of the cylinder \mathcal{D} are

$$w(x, z, t) = 0, \quad \text{on } \Gamma_z \times (0, T), \quad \text{and} \quad \int_0^h \mathbf{v}(x, z, t) \, dz \cdot n = 0, \quad \text{on } \Gamma_x \times (0, T), \tag{13}$$

where n is the outward unit normal to $\partial\mathcal{M}$. Note that there is no evolution equation for w . Instead, the incompressibility condition implies that $w(x, z, t) = -\int_0^z \operatorname{div} \mathbf{v}(x, \zeta, t) \, d\zeta$, for all $0 < z < h$, and $0 < t < T$, which combined with (13) shows that the vertical average of $\operatorname{div} \mathbf{v}$ is zero, i.e.,

$$\int_0^h \operatorname{div} \mathbf{v}(x, z, t) \, dz = 0, \tag{14}$$

for all $x \in \mathcal{M}$ and $0 < t < T$. We consider real-analytic initial data $\mathbf{v}(x, z, 0) = \mathbf{v}_0(x, z)$, and $\rho(x, z, 0) = \rho_0(x, z)$ in \mathcal{D} , which satisfy the compatibility conditions arising from (13) and (14).

The local well-posedness of the hydrostatic Euler equations is an outstanding open problem (cf. [8]). The main difficulties are to find a well-posed set of boundary conditions (cf. [16]), and to deal with the loss of derivatives which prevents the estimates to close in Sobolev spaces (cf. [4,13]). The only local existence result available for the nonlinear problem was obtained in 2D by Brenier [3] under the assumptions of convexity in the z -variable and of periodicity in the x -variable. Progress in the linearized case with a nonlocal boundary condition has been achieved in [14,15].

In the present Note we introduce the side-boundary condition (13), and prove the existence and uniqueness of solutions to (8)–(13) in the 2D case (that is with \mathcal{M} and (\mathbf{u}, ρ, p) independent of x_2), and the 3D cases where \mathcal{M} is a half-plane or the periodic domain. The solution we construct is real-analytic (up to the boundary) and is unique in this class. To the best of our knowledge this is the first local well-posedness result for the hydrostatic Euler equations in 3D, and in the absence of convexity, even in 2D. For results on analyticity of solutions to the dissipative Prandtl boundary layer equations cf. [11,17], and for the classical Euler equations cf. [1,2,5,6,10].

2. Main theorems

In the following, $\alpha = (\alpha_1, \alpha_2, \alpha_z) \in \mathbb{N}^3$ denotes a multi-index, where $\mathbb{N} = \{0, 1, 2, \dots\}$ is the set of all non-negative integers. The notation $\partial^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_z^{\alpha_z} = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_z^{\alpha_z}$, where $\alpha' = (\alpha_1, \alpha_2)$ will be used throughout. We denote the homogeneous Sobolev semi-norms $|\cdot|_m$, for $m \in \mathbb{N}$, by $|\mathbf{v}|_m = \sum_{|\alpha|=m} \|\partial^\alpha \mathbf{v}\|_{L^2(\mathcal{D})}$. In the 2D case we have $\mathcal{M} = (0, 1)$ and $\mathcal{D} = (0, 1) \times (0, h)$. For $r \geq 0$ and $\tau > 0$ fixed, define the spaces of real-analytic functions

$$X_\tau = \left\{ \mathbf{v} \in C^\infty(\mathcal{D}): \int_0^h \mathbf{v}|_{\Gamma_x} \, dz \cdot n = 0, \int_0^h \operatorname{div} \mathbf{v} \, dz = 0, \|\mathbf{v}\|_{X_\tau} = \sum_{m=0}^\infty |\mathbf{v}|_m \frac{(m+1)^r \tau^m}{m!} < \infty \right\} \tag{15}$$

and similarly, denote $Y_\tau = \{\mathbf{v} \in X_\tau, \|\mathbf{v}\|_{Y_\tau} = \sum_{m=1}^\infty |\mathbf{v}|_m (m+1)^r \tau^{m-1} / (m-1)! < \infty\}$. We write $\rho \in X_\tau$ if $\rho \in C^\infty(\mathcal{D})$ and $\|\rho\|_{X_\tau} < \infty$, and $\rho \in Y_\tau$ if $\rho \in X_\tau$ and $\|\rho\|_{Y_\tau} < \infty$. Let $\|(\mathbf{v}, \rho)\|_{X_\tau} = \|\mathbf{v}\|_{X_\tau} + \|\rho\|_{X_\tau}$, and $\|(\mathbf{v}, \rho)\|_{Y_\tau} = \|\mathbf{v}\|_{Y_\tau} + \|\rho\|_{Y_\tau}$. Using the Sobolev embedding theorem it is clear from (15) that if $\mathbf{v} \in X_\tau$ then \mathbf{v} is real-analytic with radius of analyticity τ . Conversely, if \mathbf{v} is real-analytic with radius τ (and satisfies the boundary conditions), then $\mathbf{v} \in X_{\tau'}$ for any $\tau' < \tau$ and $r \geq 0$. The following is our main result for the 2D case.

Theorem 2.1. *Let the functions \mathbf{u}, p, ρ be independent of x_2 , and let $r \geq 2$. Assume that $\mathbf{v}_0, \rho_0 \in X_{\tau_0}$, for some $\tau_0 > 0$, and suppose that \mathbf{v}_0 satisfies the compatibility conditions arising from (13) and (14). Then there exists $T_* = T_*(r, g, \tau_0, \|(\mathbf{v}_0, \rho_0)\|_{X_{\tau_0}}) > 0$, and a unique real-analytic solution $(\mathbf{v}(\cdot, t), \rho(\cdot, t))$ of the initial value problem associated with (8)–(13) with radius of analyticity $\tau(t)$, such that*

$$\begin{aligned} & \|(\mathbf{v}(t), \rho(t))\|_{X_{\tau(t)}} + Cg \int_0^t e^{g(t-s)} \|(\mathbf{v}(s), \rho(s))\|_{Y_{\tau(s)}} \, ds \\ & + C \|(\mathbf{v}_0, \rho_0)\|_{X_{\tau_0}} e^{gt} \int_0^t (1 + \tau^{-2}(s)) \|(\mathbf{v}(s), \rho(s))\|_{Y_{\tau(s)}} \, ds \leq \|(\mathbf{v}_0, \rho_0)\|_{X_{\tau_0}} e^{gt}, \end{aligned} \tag{16}$$

for all $t \in [0, T_*)$, where $C = C(\mathcal{D})$ is a fixed positive constant. Moreover, the radius of analyticity of the solution, $\tau : [0, T_*) \mapsto \mathbb{R}_+$, may be computed explicitly from (29) below.

In the 3D case, the boundary condition (13) allows us to find the pressure implicitly as the solution of an elliptic Neumann problem (for details, see [18]). The following theorem is our main result in 3D:

Theorem 2.2. *Let $r \geq 5/2$, and let \mathcal{M} be either the upper half-plane $\{x_1 > 0\}$ or the periodic domain $[0, 2\pi]^2$. Assume that $\mathbf{v}_0, \rho_0 \in X_{\tau_0}$, for some $\tau_0 > 0$, and suppose that \mathbf{v}_0 satisfies the compatibility conditions arising from (13) and (14). Then there exists $T_* = T_*(r, f, g, \tau_0, \|(\mathbf{v}_0, \rho_0)\|_{X_{\tau_0}}) > 0$, and a unique real-analytic solution $(\mathbf{v}(\cdot, t), \rho(\cdot, t))$ of the initial value problem associated with (8)–(13) with radius of analyticity $\tau(t)$, such that*

$$\begin{aligned} & \|(\mathbf{v}(t), \rho(t))\|_{X_{\tau(t)}} + Cg \int_0^t e^{C_1(t-s)} \|(\mathbf{v}(s), \rho(s))\|_{Y_{\tau(s)}} \, ds \\ & + C \|(\mathbf{v}_0, \rho_0)\|_{X_{\tau_0}} e^{C_1 t} \int_0^t (1 + \tau^{-5/2}(s)) \|(\mathbf{v}(s), \rho(s))\|_{Y_{\tau(s)}} \, ds \leq \|(\mathbf{v}_0, \rho_0)\|_{X_{\tau_0}} e^{C_1 t}, \end{aligned} \tag{17}$$

for all $t \in [0, T_*)$, where $C = C(\mathcal{D})$ and $C_1 = C_1(f, g)$ are fixed positive constants. Moreover, the radius of analyticity of the solution $\tau : [0, T_*) \mapsto \mathbb{R}_+$ may be computed explicitly.

3. The two-dimensional case

In this section we give the *a priori* estimates needed to prove Theorem 2.1. These estimates will be made rigorous in Section 4. Let $r \geq 2$ be fixed throughout the rest of the section, and assume that $(\mathbf{v}, w), p$, and ρ are x_2 -independent smooth solutions of (8)–(13), with $\mathbf{v}_0, \rho_0 \in X_{\tau_0}$, for some $\tau_0 > 0$. We denote $\partial_x = \partial_{x_1}$, and $\Delta = \partial_{x_1 x_1}$. It is convenient to write (8)–(13) in component form

$$\partial_t v_1 + v_1 \partial_x v_1 + w \partial_z v_1 + \partial_x p + f v_2 = g \partial_x \psi, \tag{18}$$

$$\partial_t v_2 + v_1 \partial_x v_2 + w \partial_z v_2 - f v_1 = 0, \tag{19}$$

$$\partial_t \rho + v_1 \partial_x \rho + w \partial_z \rho = 0, \tag{20}$$

where $\psi(x, z) = \int_0^z \rho(x, \zeta) \, d\zeta$. Additionally, $\partial_x v_1 + \partial_z w = 0$ and $\partial_z p = 0$ hold. The boundary conditions for w and v_1 are $w = 0$ on Γ_z , and $\int_0^h v_1 \, dz = 0$ on Γ_x , where $\Gamma_x = \{0, 1\}$. We note that there is no boundary condition for v_2 . Integrating the incompressibility condition in z , and using the boundary conditions we obtain

$$\int_0^h v_1(x, z) \, dz = 0, \quad \text{for all } x \in \mathcal{M}. \tag{21}$$

The boundary condition for w , and identity (21) give (cf. [7]) the pressure explicitly as

$$p(x) = - \int_0^h v_1^2(x, z) dz - f \int_0^{x_1} \int_0^h v_2(x'_1, x_2, z) dz dx'_1 + g \int_0^h \psi(x, z) dz. \tag{22}$$

Also, we have the cancellation property $\langle \partial_x \partial^\alpha p, \partial^\alpha v_1 \rangle = 0$, for any multi-index $\alpha \in \mathbb{N}^3$. In (22) and in the following we use the notation $\int_0^h \phi(x, z) dz = (1/h) \int_0^h \phi(x, z) dz$, for any smooth function ϕ .

Proof of Theorem 2.1. From (15) it follows that

$$\frac{d}{dt} \|\mathbf{v}(t)\|_{X_{\tau(t)}} = \dot{\tau}(t) \|\mathbf{v}(t)\|_{Y_{\tau(t)}} + \sum_{m=0}^{\infty} \sum_{|\alpha|=m} \frac{d}{dt} \|\partial^\alpha \mathbf{v}(t)\|_{L^2} \frac{(m+1)^r \tau(t)^m}{m!}. \tag{23}$$

Given a multi-index $\alpha \in \mathbb{N}^2$, we estimate $(d/dt) \|\partial^\alpha \mathbf{v}(t)\|_{L^2}$ by applying ∂^α to (18)–(19), taking the $L^2(\mathcal{D})$ -inner product $\langle \cdot, \cdot \rangle$ with $\partial^\alpha \mathbf{v}$, and using $\langle \partial^\alpha v^\perp, \partial^\alpha v \rangle = 0$. The pressure term arising in this process vanishes since $\langle \partial_x \partial^\alpha p, \partial^\alpha v_1 \rangle = 0$. Upon summing over $\alpha \in \mathbb{N}^2$, the Schwarz inequality and (23) give

$$\frac{d}{dt} \|\mathbf{v}\|_{X_\tau} \leq \dot{\tau} \|\mathbf{v}\|_{Y_\tau} + \mathcal{U}(\mathbf{v}, \mathbf{v}) + \mathcal{V}(w, \mathbf{v}) + g \|\partial_x \psi\|_{X_\tau}, \tag{24}$$

where for vector functions $\mathbf{v}, \tilde{\mathbf{v}} \in X_\tau$, we have denoted

$$\mathcal{U}(\mathbf{v}, \tilde{\mathbf{v}}) = \sum_{m=0}^{\infty} \sum_{j=0}^m \sum_{|\alpha|=m} \sum_{|\beta|=j, \beta \leq \alpha} \binom{\alpha}{\beta} \|\partial^\beta \mathbf{v} \cdot \nabla \partial^{\alpha-\beta} \tilde{\mathbf{v}}\|_{L^2} \frac{(m+1)^r \tau^m}{m!}, \tag{25}$$

and

$$\mathcal{V}(w, \tilde{\mathbf{v}}) = \sum_{m=0}^{\infty} \sum_{j=0}^m \sum_{|\alpha|=m} \sum_{|\beta|=j, \beta \leq \alpha} \binom{\alpha}{\beta} \|\partial^\beta w \partial_z \partial^{\alpha-\beta} \tilde{\mathbf{v}}\|_{L^2} \frac{(m+1)^r \tau^m}{m!}. \tag{26}$$

In (26) we let $w(x, z) = - \int_0^z \operatorname{div} \mathbf{v}(x, \zeta) d\zeta$. The convection terms \mathcal{U} and \mathcal{V} are estimated as

$$\mathcal{U}(\mathbf{v}, \tilde{\mathbf{v}}) + \mathcal{V}(w, \tilde{\mathbf{v}}) \leq C_0(1 + \tau^{-2}) \|\mathbf{v}\|_{X_\tau} \|\tilde{\mathbf{v}}\|_{Y_\tau} + C_0(1 + \tau^{-1}) \|\tilde{\mathbf{v}}\|_{X_\tau} \|\mathbf{v}\|_{Y_\tau}, \tag{27}$$

for some sufficiently large positive constant $C_0 = C_0(\mathcal{D})$. To bound the last term on the right side of (24) we note that by (11) we have $|\partial_x \psi|_m \leq C_0 \rho|_{m+1} + \rho|_m$, and therefore $g \|\partial_x \psi\|_{X_\tau} \leq C_0 g \|\rho\|_{Y_\tau} + g \|\rho\|_{X_\tau}$. We fix $C_0 = C_0(\mathcal{D})$ as in (27). The growth of $\|\rho\|_{X_\tau}$ is controlled by $(d/dt) \|\rho\|_{X_\tau} \leq \dot{\tau} \|\rho\|_{Y_\tau} + \mathcal{U}(\mathbf{v}, \rho) + \mathcal{V}(w, \rho)$, and the right side of this estimate is bounded similarly to (27). After some computations we obtain

$$\frac{d}{dt} \|(\mathbf{v}, \rho)\|_{X_\tau} \leq (\dot{\tau} + C_0 g + 3C_0(1 + \tau^{-2})) \|(\mathbf{v}, \rho)\|_{X_\tau} \|(\mathbf{v}, \rho)\|_{Y_\tau} + g \|(\mathbf{v}, \rho)\|_{X_\tau}. \tag{28}$$

Define a decreasing function $\tau(t)$ by

$$\dot{\tau} + 20C_0 g + 20C_0(1 + \tau^{-2}) \|(\mathbf{v}_0, \rho_0)\|_{X_{\tau_0}} e^{g\tau} = 0, \tag{29}$$

and $\tau(0) = \tau_0$; this uniquely determines τ in terms of the initial data. Let T_* be the maximal time such that $\tau(t) \geq 0$. By construction, we have $\|(\mathbf{v}(t), \rho(t))\|_{X_{\tau(t)}} \leq \|(\mathbf{v}_0, \rho_0)\|_{X_{\tau_0}} e^{g\tau}$ for all $t < T_*$, and by (28), we obtain that the solution is *a priori* bounded in $L^\infty(0, T_*; X_{\tau}) \cap L^1(0, T_*; (1 + \tau^{-2}) Y_\tau)$ in the sense that (16) holds for all $t < T_*$, with $C = 10C_0$. See [7] for details. This completes the proof of the theorem. \square

4. Construction of the solution and uniqueness for the two-dimensional case

The formal construction of the solution is via the Picard iteration. Let $\mathbf{v}^{(0)} = \mathbf{v}_0$ and $\rho^{(0)} = \rho_0$, where \mathbf{v}_0 satisfies the compatibility conditions arising from (13) and (14). For $n \in \mathbb{N}$, we define $w^{(n)}(x, z, t) = - \int_0^z \operatorname{div} \mathbf{v}^{(n)}(x, \zeta, t) d\zeta$ and $\psi^{(n)}(x, z, t) = \int_0^z \rho^{(n)}(x, \zeta, t) d\zeta$. The density iterate is

$$\rho^{(n+1)}(t) = \rho_0 - \int_0^t (\mathbf{v}^{(n)}(s) \cdot \nabla + w^{(n)}(s) \partial_z) \rho^{(n)}(s) ds, \tag{30}$$

and using (22) we define the pressure iterate by

$$p^{(n+1)}(x, t) = - \int_0^h (v_1^{(n)})^2(x, z, t) dz - f \int_0^{x_1} \int_0^h v_2^{(n)}(x'_1, x_2, z, t) dz dx'_1 + g \int_0^h \psi^{(n+1)}(x, z, t) dz. \tag{31}$$

Lastly, the velocity iterate is constructed as

$$\mathbf{v}^{(n+1)}(t) = \mathbf{v}_0 - \int_0^t (\mathbf{v}^{(n)} \cdot \nabla + w^{(n)} \partial_z) \mathbf{v}^{(n)}(s) + \nabla p^{(n+1)}(s) - g \nabla \psi^{(n+1)}(s) + f \mathbf{v}^{(n)\perp}(s) ds, \tag{32}$$

for all $n \in \mathbb{N}$. By construction, the compatibility conditions $\int_0^h \operatorname{div} v_1^{(n)} dz = 0$ and the boundary condition $\int_0^h v_1^{(n)}|_{\Gamma_x} dz = 0$ are conserved for all $n \in \mathbb{N}$. We define $\tau(t)$ by $\tau(0) = \tau_0$ and $\dot{\tau}(t) + 20C_0g + 20C_0(1 + \tau^{-2}(t))\|(\mathbf{v}_0, \rho_0)\|_{X_{\tau_0}} e^{gt} = 0$, where the constant $C_0 = C_0(r, \mathcal{D})$ is fixed. The sequence of approximations $\mathbf{v}^{(n)}$ is bounded in $L^\infty(0, T; X_\tau) \cap L^1(0, T; (1 + \tau^{-2})Y_\tau)$ for some sufficiently small $T > 0$, and moreover, the map $\mathbf{v}^{(n)} \mapsto \mathbf{v}^{(n+1)}$ is a contraction in $L^\infty(0, T; X_\tau) \cap L^1(0, T; (1 + \tau^{-2})Y_\tau)$ (cf. [7]).

In order to prove the uniqueness, assume that there exist two solutions $(\mathbf{v}^{(i)}, \rho^{(i)})$, $i = 1, 2$, to (8)–(13) evolving from initial data (\mathbf{v}_0, ρ_0) , with $(\mathbf{v}^{(i)}, \rho^{(i)}) \in L^\infty(0, T; X_\tau) \cap L^1(0, T; (1 + \tau^{-2})Y_\tau)$ for $i = 1, 2$. We denote the difference of the solutions by $\mathbf{v} = \mathbf{v}^{(1)} - \mathbf{v}^{(2)}$ and $\rho = \rho^{(1)} - \rho^{(2)}$. Similarly to the *a priori* estimates of Section 3, we obtain

$$\begin{aligned} & \frac{d}{dt} \|(\mathbf{v}, \rho)\|_{X_\tau} + 10C_0g \|(\mathbf{v}, \rho)\|_{Y_\tau} + 10C_0 \|(\mathbf{v}_0, \rho_0)\|_{X_{\tau_0}} e^{gt} (1 + \tau^{-2}) \|(\mathbf{v}, \rho)\|_{Y_\tau} \\ & \leq g \|(\mathbf{v}, \rho)\|_{X_\tau} + 3C_0(1 + \tau^{-2})(\|\mathbf{v}^{(1)}\|_{Y_\tau} + \|\mathbf{v}^{(2)}\|_{Y_\tau}) \|(\mathbf{v}, \rho)\|_{X_\tau}. \end{aligned} \tag{33}$$

It is straightforward to check that (33), the *a priori* bound on $\int_0^t (1 + \tau^{-2}) \|\mathbf{v}^{(i)}\|_{Y_\tau} ds$, and Grönwall's inequality imply that $\|(\mathbf{v}, \rho)\|_{X_\tau} = 0$ for all $t \in [0, T_*)$, thereby proving the uniqueness of solutions.

5. The three-dimensional case

In the periodic case we have $\mathcal{M} = [0, 2\pi]^2$, and the boundary condition (13) is replaced by the periodic boundary condition in the x -variable. Similarly to estimate (24), we have $(d/dt)\|\mathbf{v}(t)\|_{X_{\tau(t)}} \leq \dot{\tau}(t)\|\mathbf{v}(t)\|_{Y_{\tau(t)}} + \mathcal{U}(\mathbf{v}, \mathbf{v}) + \mathcal{V}(w, \mathbf{v}) + \mathcal{P} + g\|\nabla\psi\|_{X_\tau}$, where $\mathcal{U}(\mathbf{v}, \mathbf{v})$ and $\mathcal{V}(w, \mathbf{v})$ are defined by (25) and (26) respectively, while

$$\mathcal{P} = \sum_{m=1}^{\infty} \sum_{|\alpha|=m} \|\nabla \partial^\alpha p\|_{L^2(\mathcal{D})} \frac{(m+1)^r \tau^m}{m!} = h^{1/2} \sum_{m=1}^{\infty} \sum_{|\alpha|=m, \alpha_3=0} \|\nabla \partial^\alpha p\|_{L^2(\mathcal{M})} \frac{(m+1)^r \tau^m}{m!}. \tag{34}$$

To estimate \mathcal{P} , we use the fact that the pressure may be computed explicitly from the velocity as

$$p = R_j R_k \int_0^h v_j v_k dz + f(-\Delta)^{-1/2} \int_0^h (R_1 v_2 - R_2 v_1) dz + g \int_0^h \psi dz, \tag{35}$$

where R_j is the j th Riesz transform. The proof of Theorem 2.2 follows in analogy to the 2D case (cf. Section 3). The construction of the x -periodic solution is similar to Section 4, but instead of $p^{(n)}$ being defined by (31), we define the n th iterate of the pressure to respect (35).

For the case \mathcal{M} is a half-plane, let $\mathcal{M} = \{x \in \mathbb{R}^2: x_1 > 0\}$, so that the side-boundary condition (13) is $\int_0^h v_1(0, x_2, z) dz = 0$. The main difficulty is to bound the pressure term \mathcal{P} . This is achieved by noting that the vertical average of the full pressure $\bar{p}(x) = \int_0^h P(x, z) dz = p(x) - g \int_0^h \psi(x, z) dz$ satisfies

$$-\Delta \bar{p} = \partial_k \int_0^h (v_j \partial_j v_k + v_k \partial_j v_j) dz + f \int_0^h (\partial_1 v_2 - \partial_2 v_1) dz, \tag{36}$$

$$\frac{\partial \bar{p}}{\partial n} = \int_0^h (v_1 \partial_j v_j + v_j \partial_j v_1) dz + f \int_0^h v_2 dz, \tag{37}$$

where $j, k \in \{1, 2\}$. We refer the reader to [7] for details.

Acknowledgements

The work of I.K. and V.V. was supported in part by the NSF grant DMS-0604886, the work of R.T. was partially supported by the National Science Foundation under the grant NSF-DMS-0906440 and by the Research Fund of Indiana University, while M.Z. was supported in part by the NSF grant DMS-0505974.

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