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Zhang's conjecture and squares of Abelian surfaces

Conjecture de Zhang et carrés de surfaces abéliennes

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ABSTRACT

We give in this Note some squares of Abelian surfaces that are counterexamples to a conjecture formulated by Zhang about the intersection of subvarieties and preperiodic points.

RÉSUMÉ

On donne dans cette Note des exemples de carrés de surfaces abéliennes violant la conclusion de la conjecture de Zhang sur l'intersection des sous-variétés et des points prépériodiques.

1. Introduction

D. Ghioca and T. Tucker found a family of counterexamples to Zhang's dynamical Manin–Mumford Conjecture 1.2.1 of [5]. They use squares of elliptic curves with complex multiplication. D. Ghioca asked whether this counterexample could be generalized. We present here counterexamples of greater dimension. We recall a few definitions: an endomorphism $\varphi : X \to X$ of a projective variety is said to have a *polarization* if there exists an ample divisor D such that $\varphi^* D \sim dD$ for some $d > 1$, where $\sim$ stands for the linear equivalence. A subvariety Y of X is *preperiodic* under φ if there exist integers $m \geqslant 0$ and $k > 0$ such that $\varphi^{m+k}(Y) = \varphi^m(Y)$. We denote $\mathrm{Prep}_\varphi(X)$ the set of preperiodic points of X under the action of φ. We now recall the conjecture:

Conjecture 1.1 *(Algebraic dynamical Manin–Mumford). Let $\varphi : X \to X$ be an endomorphism of a projective variety defined over a number field K with a polarization, and let Y be a subvariety of X. If $Y \cap \mathrm{Prep}_\varphi(X)$ is Zariski-dense in Y, then Y is a preperiodic subvariety.*

We will use the following lemma of Ghioca and Tucker [1]:

Lemma 1.2. *Let A be a simple Abelian variety and $\varphi_1, \varphi_2 \in \mathrm{Isog}(A)$ be nonzero. Let $m \geqslant 0$ and $k > 0$ be two integers. Let $\Delta = \{(x, x) \mid x \in A\}$. Then $(\varphi_1^m, \varphi_2^m)(\Delta) \subset (\varphi_1^{m+k}, \varphi_2^{m+k})(\Delta)$ if and only if $\varphi_1^k = \varphi_2^k$.*

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In a nutshell, for the examples that we find in this Note, Δ is preperiodic under $\varphi = (\varphi_1, \varphi_2)$ if and only if the action of φ_1 and φ_2 differs by a root of unity.

Proof. For the direct part, one checks that $\varphi_1^k - \varphi_2^k$ sends nontorsion points to torsion points, hence is the zero map. The converse comes from the surjectivity of φ_1 and φ_2. $\square$

The examples we provide are inspired by Ghioca and Tucker's original ones, the main problem to overcome is finding a polarizable situation. The idea used here is the theorem of the cube on Abelian varieties combined with some particular properties of the field of definition.

For other reflections on the dynamical Manin–Mumford conjecture, one can refer to [1] or [4].

2. Polarizability criterion

We give in this section a few formulas useful to get information on the weight of complex multiplication. We start with a general fact:

Proposition 2.1. *Let A be an Abelian variety, V a variety and f, g, h three morphisms from V to A. Then for any divisor $D \in \mathrm{Div}(A)$, one has*

$$(f+g+h)^*D - (f+g)^*D - (g+h)^*D - (f+h)^*D + f^*D + g^*D + h^*D \sim 0.$$

Proof. This statement is a direct consequence of the theorem of the cube. For a proof, see for example [2, Corollary A.7.2.4, p. 123]. $\square$

Let A be an Abelian variety and suppose it has complex multiplication by a ring R, *i.e.* the ring of endomorphisms of A contains R and R contains strictly $\mathbb{Z}$ (see [3, Chapter II.1], for the case of elliptic curves). Then we have the following lemma:

Lemma 2.2. *Let A be an Abelian variety and let D be a divisor on A. Let $n \in \mathbb{N}$ and $\alpha \in R$. Then*

$$[n+\alpha]^*D \sim n[1+\alpha]^*D - (n-1)[\alpha]^*D + n(n-1)D. \tag{1}$$

Proof. Use Proposition 2.1 with $f = [n-1]$, $g = [\alpha]$ and $h = [1]$. The result follows by a recurrence and a telescoping sum. $\square$

Corollary 2.3. *If one chooses D such that $[\alpha]^*D \sim D$, then $[n+\alpha]$ is polarized by D if and only if $[1+\alpha]$ is polarized by D and one has*

$$[n+\alpha]^*D \sim n[1+\alpha]^*D + (n-1)^2 D. \tag{2}$$

3. Theta divisor in dimension 2

Let C be a curve of genus 2 defined over $\bar{\mathbb{Q}}$. Choose an affine equation $y^2 = f(x)$ with $\deg(f) = 5$, and let ∞ be the point at infinity. Let $\mathrm{Jac}(C)$ denote the Jacobian of C. We denote by $cl(D)$ the linear equivalence class of any divisor D on $\mathrm{Jac}(C)$. Let $\Theta = j(C)$ be the theta divisor, where

$$j : C \hookrightarrow \mathrm{Jac}(C),$$
$$P \to cl\big((P) - (\infty)\big).$$

Consider the surjective map

$$\mathrm{Sym}^2(C) \to \mathrm{Jac}(C),$$
$$\{P_1, P_2\} \to cl\big((P_1) + (P_2) - 2(\infty)\big).$$

Take $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$. Then the hyperelliptic involution $\iota : (x, y) \to (x, -y)$ gives the multiplication by $[-1]$ on $\mathrm{Jac}(C)$. We have to blow down all the points $\{P, \iota(P)\}$ on $\mathrm{Sym}^2(C)$ to the origin of $\mathrm{Jac}(C)$. The theta divisor is then the image of the set of all pairs $\{P, \infty\}$ with $P \in C$. It is symmetric and ample.

If one chooses the affine equation to be $y^2 = f(x)$ with $\deg(f) = 6$, then there are two points at infinity ∞^+ and ∞^-, and over a field extension one gets a point ∞ such that $(\infty^+) + (\infty^-) \sim 2(\infty)$. Then one would work with the divisor $D_0 = t^*_{\infty-\infty^+}\Theta + t^*_{\infty-\infty^-}\Theta \sim 2\Theta$, where t_P stands for the translation by the point P.

4. Example in degree 5

Let us focus on the curve C with affine model $y^2 = x^5 - x$. In this particular case, we get a Jacobian with complex multiplication, coming from $[i] : (x, y) \to (-x, iy)$, where $i^2 = -1$.

We have $[i]^*(x, y) = (-x, -iy)$ on the curve, which gives $[i]^*(\{(x, y), \infty\}) = \{(-x, -iy), \infty\}$ on $\mathrm{Sym}^2(C)$, thus $[i]^*\Theta \sim \Theta$. Let us use Proposition 2.1 in the following situation: $A = V = \mathrm{Jac}(C)$, $D = \Theta$, $f = [i]$, $g = [1]$ and $h = [-1]$. Then we get

$$[i]^*\Theta - [1+i]^*\Theta - [i-1]^*\Theta + [i]^*\Theta + \Theta + [-1]^*\Theta \sim 0,$$

thus using $[-1]^*\Theta \sim \Theta$ and $[i]^*\Theta \sim \Theta$ we have

$$[1+i]^*\Theta + [1-i]^*\Theta \sim 4\Theta. \tag{3}$$

Let us now remark that $i(1-i) = 1+i$, so $[1+i]^*\Theta \sim [i(1-i)]^*\Theta \sim [1-i]^*[i]^*\Theta \sim [1-i]^*\Theta$. Using this in Eq. (3), one gets

$$[1+i]^*\Theta \sim 2\Theta. \tag{4}$$

Then using Corollary 2.3 one gets $[2+i]^*\Theta \sim 5\Theta$ and $[2-i]^*\Theta \sim 5\Theta$.

Let $A = \mathrm{Jac}(C)$ and let $\varphi = [2+i] \times [2-i]$. Consider the following situation

$$\varphi : A \times A \to A \times A.$$

The morphism φ is polarized by $D = \pi_1^*\Theta + \pi_2^*\Theta$, where π_1 and π_2 are respectively the first and second projections, and $\varphi^*D \sim 5D$. Choose $Y = \Delta = \{(P, P) \in A \times A\}$ to be the diagonal. The intersection $Y \cap \mathrm{Prep}(A \times A)$ is Zariski-dense in Y. Then $\varphi^k Y = Y$ implies that for every $P \in Y$ we have $[2-i]^k P = [2+i]^k P$. But $\frac{2-i}{2+i}$ is not a root of unity. Use Lemma 1.2. We thus have provided a square of an Abelian surface that contradicts Conjecture 1.1.

5. Example in degree 6

Let us focus on the curve C with affine model $y^2 = x^6 - 1$. With this choice of affine model, one has two points at infinity denoted ∞^+ and ∞^-. We consider the endomorphism $[\alpha] : (x, y) \to (\alpha x, y)$, where $\alpha^6 = 1$. This morphism gives rise to a complex multiplication endomorphism on the surface $\mathrm{Jac}(C)$ that will also be denoted $[\alpha]$. The divisor $(\infty^+) + (\infty^-)$ is invariant under $[\alpha]$. We split the study into two cases, whether we have $\alpha = j$ where $j^2 + j + 1 = 0$ or $\alpha^2 - \alpha + 1 = 0$. We begin with $\alpha = j$. Define the divisor

$$D_1 = D_0 + [j]^*D_0 + \left[j^2\right]^*D_0 + [-1]^*D_0 + [-j]^*D_0 + \left[-j^2\right]^*D_0.$$

One verifies that $[\pm j^m]^*D_1 \sim D_1$ for $m = 0, 1, 2$. Let us use Proposition 2.1 in the following situation: $A = V = \mathrm{Jac}(C)$, $D = D_1$, $f = [1]$, $g = [j]$ and $h = [j]$. Then we get

$$[1+2j]^*D_1 - 2[1+j]^*D_1 - [2j]^*D_1 + D_1 + 2[j]^*D_1 \sim 0,$$

thus using $1 + 2j = j - j^2$ and $1 + j = -j^2$, plus $[-1]^*D_1 \sim D_1$ and $[j]^*D_1 \sim D_1$, we have

$$[1-j]^*D_1 \sim 3D_1. \tag{5}$$

Let us now remark that $(1-j^2)j = j - 1$, so $[1-j]^*D_1 \sim [j(1-j^2)]^*D_1 \sim [1-j^2]^*[j]^*D_1 \sim [1-j^2]^*D_1$. Using this in Eq. (5), one gets

$$\left[1-j^2\right]^*D_1 \sim 3D_1. \tag{6}$$

Then by using (5) and Corollary 2.3 one gets $[2-j]^*D_1 \sim 7D_1$, and by using (6) and Corollary 2.3 one gets $[2-j^2]^*D_1 \sim 7D_1$.

Let $A = \mathrm{Jac}(C)$ and let $\varphi = [2-j] \times [2-j^2]$. Consider the following situation

$$\varphi : A \times A \to A \times A.$$

The morphism φ is polarized by $D = \pi_1^*D_1 + \pi_2^*D_1$, where π_1 and π_2 are respectively the first and second projections, and $\varphi^*D \sim 7D$. Choose $Y = \Delta = \{(P, P) \in A \times A\}$ to be the diagonal. The intersection $Y \cap \mathrm{Prep}(A \times A)$ is Zariski-dense in Y. Then $\varphi^k Y = Y$ implies that for every $P \in Y$ we have $[2-j]^k P = [2-j^2]^k P$. But $\frac{2-j}{2-j^2}$ is not a root of unity.

One may deal with the case $\alpha^2 = \alpha - 1$ in the same way, using Proposition 2.1 with $f = [1]$, $g = h = [-\alpha]$.

6. Multiplication by ζ_5 not polarized by Θ

Let us focus on the curve C with affine model $y^2 = x^5 - 1$. In this particular case, we get a Jacobian with complex multiplication coming from $[\zeta_5] : (x, y) \to (\zeta_5 x, y)$, where $\zeta_5^5 = 1$. We have $[\zeta_5]^*(x, y) = (\zeta_5^4 x, y)$ on the curve, which gives $[\zeta_5]^*(\{(x, y), \infty\}) = \{(\zeta_5^4 x, y), \infty\}$ on $\mathrm{Sym}^2(C)$, thus $[\zeta_5]^*\Theta \sim \Theta$. We gather a few pullback formulas in this particular setting:

Lemma 6.1. *Let m and n be integers. One has*

$$\left[n + \zeta_5^m\right]^*\Theta + \left[n - \zeta_5^m\right]^*\Theta \sim \left(2n^2 + 2\right)\Theta, \tag{7}$$

$$[1 + \zeta_5]^*\Theta + \left[1 + \zeta_5^2\right]^*\Theta \sim 3\Theta, \tag{8}$$

$$\left[(1 + \zeta_5)\left(1 + \zeta_5^2\right)\right]^*\Theta \sim \Theta. \tag{9}$$

Proof. The first equality can be deduced from (2), using $\alpha = \zeta_5^m$ and $\alpha = -\zeta_5^m$. The second equality comes from the application of Proposition 2.1 with $f = [1]$, $g = [\zeta_5]$ and $h = [\zeta_5^2]$. The last equality comes from the relation $1 + \zeta_5 + \zeta_5^2 + \zeta_5^3 = -\zeta_5^4$ and the fact that $[\zeta_5]^*\Theta \sim \Theta$. $\square$

As opposed to the endomorphisms $[i]$ and $[j]$ in the first examples, the formulas (8) and (9) show that $[\zeta_5]$ will not be polarized by Θ.

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