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Ordinary Differential Equations

A theorem of uniqueness for an inviscid dyadic model

Un théorème d'unicité pour un modèle dyadique non visqueux

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ABSTRACT

We consider the solutions of the Cauchy problem for a dyadic model of Euler equations. We prove global existence and uniqueness of Leray–Hopf solutions in a rather large class $\mathcal{K}$ that implies in particular global existence and uniqueness in l^2 for all initial positive conditions in l^2.

RÉSUMÉ

Nous considérons les solutions du problème de Cauchy pour un modèle dyadique d'équations d'Euler. Nous démontrons l'existence et l'unicité globales des solutions de Leray–Hopf dans une classe $\mathcal{K}$ assez large, ce qui implique en particulier l'existence et l'unicité dans l^2 pour toute condition initiale positive dans l^2.

1. Introduction

This Note is concerned with the following infinite system of differential equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}X_n(t) = k_{n-1}X_{n-1}^2(t) - k_nX_n(t)X_{n+1}(t), \quad t \geqslant 0,\ X_n(0) = x_n, \tag{1}$$

for $n \geqslant 1$, with $0 \leqslant k_n \leqslant C2^n$ for every $n \geqslant 1$, $k_0 = 0$ and $X_0(t) = 0$ for every $t \geqslant 0$. This system is usually called *dyadic model of turbulence*. It has been introduced by Kats and Pavlovic in [6] (see also [5]) and studied in several works, including [7,9,3,1]. When the initial condition $x = (x_n)_{n\in\mathbb{N}}$ is in $H^1 = \{u = (u_n)_{n\in\mathbb{N}}\colon\ \|u\|_{H^1}^2 := \sum_n k_n^2u_n^2 < \infty\}$, a local unique solution exists in this space. However, [7] proved that such regularity is lost in finite time. After the blow-up, we can only work in the space $H = l^2$.

Global existence of solutions is known in H. By solution in H on $[0, T)$ (global if $T = +\infty$), we mean a sequence of functions $X = (X_n)_{n\in\mathbb{N}}$ defined on $[0, T)$ such that $X_n \in C^1([0, T); \mathbb{R})$ for all n, X satisfies system (1) and $X(t) \in H$ for all $t \in [0, T)$. However, uniqueness of such solutions is false, for general initial conditions $x \in H$, as shown in [1]. The aim of this note is to prove uniqueness of solutions which belong to a special class. Only certain initial conditions $x \in H$ give rise to global solutions in this class. In particular, this is true when (almost) all components of x are non-negative. Thus, the picture emerging from [1] plus the present note is that there is a large class of initial conditions with global existence and uniqueness, and examples of initial conditions with multiple solutions.

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We remark that the idea of the proof is to use cancellations, instead of estimates of functional analytic nature as it is usually done in infinite dimensions. From the functional analytic viewpoint, in order to see these cancellations, we need to work in "negative order space", namely with the expression $\sum_{i=1}^{n} 2^{-i} Z_i^2(t)$ where Z is the difference between two solutions.

The energy inequality plays a role in our uniqueness result and, moreover, it is violated by the known counterexamples to uniqueness (see [1]). Due to the importance of this issue, we end the note with a closer look to it. We prove in particular the equivalence between weak and strong forms of energy inequality.

Finally, we remark that uniqueness, for all initial conditions, is restored by adding a suitable noise, see [2]. That result partially motivated the present note and poses the following question: how large is the set of initial conditions for the deterministic system (1) which has uniqueness? Maybe the uniqueness under noise is related to the fact that non-uniqueness holds only for special initial conditions. The result of this note is a first step in this direction, but the set H_+ of initial conditions that we identify is still small.

2. The theorem of uniqueness

Let us restrict the attention to solutions which satisfy an energy inequality, as initially proposed by [8] in the viscous case. We start the discussion with the weak energy inequality and add comments in the next section on a stronger form.

Definition 2.1. A solution X on $[0, T)$ satisfies the *weak energy inequality* if

$$\left\| X(t) \right\|_H \leqslant \left\| X(0) \right\|_H \quad \forall t \in [0, T). \tag{2}$$

The existence of solutions with this property, for every $x \in H$, can be found in [1].

Definition 2.2. We call *solution of class $\mathcal{K}$* any solution $X = (X_n)_{n\in\mathbb{N}}$ such that the function

$$a(t) = \sup_{n\in\mathbb{N}} \bigl(-k_n X_{n+1}(t)\bigr) \tag{3}$$

is locally integrable on $[0, \infty)$.

Our main result is the uniqueness of solutions of class $\mathcal{K}$, satisfying (2).

Theorem 2.1. *Let $X^{(i)} = (X_n^{(i)})_{n\in\mathbb{N}}$, $i = 1, 2$, be two solutions with the same initial condition $x = (x_n)_{n\in\mathbb{N}} \in H$. Assume that $X^{(i)}$ are of class $\mathcal{K}$ and they both satisfy the weak energy inequality. Then $X^{(1)} = X^{(2)}$.*

Proof. By (2), we have

$$\left| X_n^{(i)}(t) \right| \leqslant \sqrt{E(0)}, \tag{4}$$

for all $n \geqslant 1$, $t \geqslant 0$ and $i = 1, 2$. We shall use this bound below. Let

$$Z_n := X_n^{(1)} - X_n^{(2)}, \qquad Y_n := X_n^{(1)} + X_n^{(2)}.$$

It is easy to check that for all $n \geqslant 1$, $Z_n(0) = 0$ and for $t \geqslant 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t} Z_n = k_{n-1} Z_{n-1} Y_{n-1} - \frac{k_n}{2} (Z_n Y_{n+1} + Y_n Z_{n+1}).$$

This implies

$$\frac{\mathrm{d}}{\mathrm{d}t} Z_n^2 = 2k_{n-1} Y_{n-1} Z_{n-1} Z_n - k_n Y_{n+1} Z_n^2 - k_n Y_n Z_n Z_{n+1}.$$

The terms $k_n Y_{n+1} Z_n^2$ and $k_{n+1} Y_{n+2} Z_{n+1}^2$ have a dissipative nature. The idea of the proof is to use cancellations to deal with the more difficult terms $-k_n Y_n Z_n Z_{n+1}$ and $2k_n Y_n Z_n Z_{n+1}$. But they differ by a factor -2. For this reason, instead of using the classical quantity $\sum_{i=1}^{n} Z_i^2(t)$, we introduce

$$\psi_n(t) := \sum_{i=1}^{n} 2^{-i} Z_i^2(t).$$

Indeed

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{Z_n^2}{2^n} = k_{n-1} Y_{n-1} \frac{Z_{n-1} Z_n}{2^{n-1}} - k_n Y_{n+1} \frac{Z_n^2}{2^n} - k_n Y_n \frac{Z_n Z_{n+1}}{2^n},$$

so $\psi_n(t)$ satisfies the simple equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi_n(t) = -\sum_{i=1}^{n} k_i Y_{i+1}\frac{Z_i^2}{2^i} - \frac{k_n}{2^n} Y_n Z_n Z_{n+1}, \quad \psi_n(0) = 0.$$

Since both solutions are of class $\mathcal{K}$, denoted by a the maximum of the functions $a^{(i)}$'s defined by (3), we have $-k_i Y_{i+1}(t) \leqslant 2a(t)$. Hence, using also $|Z_{n+1}| < 2\sqrt{E(0)}$, and $k_n \leqslant C2^n$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi_n(t) \leqslant 2a(t)\psi_n(t) + KY_n Z_n \leqslant 2a(t)\psi_n(t) + K\left(\left|X_n^{(1)}\right|^2 + \left|X_n^{(2)}\right|^2\right).$$

From Gronwall lemma

$$0 \leqslant \psi_n(t) \leqslant \int_0^t e^{\int_s^t 2a(r)\,\mathrm{d}r} K \sum_i \left|X_n^{(i)}(s)\right|^2 \mathrm{d}s \leqslant K'(t) \sum_i \int_0^t \left|X_n^{(i)}(s)\right|^2 \mathrm{d}s,$$

where $K'(t)$ is a positive constant for every $t \geqslant 0$. Since $\sum_n \int_0^t |X_n^{(i)}(s)|^2 = \int_0^t E^{(i)}(s)\,\mathrm{d}s \leqslant tE(0)$, both integrals above ($i = 1, 2$) tend to zero as $n \to \infty$, and $\psi_n(t)$ as well. Since the latter is non-decreasing in n, we get that $\psi_n(t) = 0$ for every n, and every $t \geqslant 0$. This implies $Z \equiv 0$. □

The essential question is now: which initial conditions give rise to global solutions satisfying the assumptions of the previous theorem? We can exhibit a relevant class, H_+, which includes the class of positive initial conditions investigated by many authors. Denote by H_+ the set of all $x \in H$ such that $x_n < 0$ for at most a finite number of n's. We can loosely say that almost all components of x are positive. Next lemma shows that this class is (positively) invariant. Denote by $S(t)x$ the set of all values at time t of solutions with initial condition x. This defines a (possibly) multivalued map $S(t): H \to \mathcal{P}(H)$, for all $t \geqslant 0$, where $\mathcal{P}(H)$ is the set of all parts of H. We may call $S(t)$ the *multivalued flow* associated to the dyadic model.

Lemma 2.2. *If X is a solution, j is a positive integer and $s \geqslant 0$, then*

$$X_j(s) \geqslant 0 \quad \Longrightarrow \quad X_j(t) \geqslant 0 \quad \textit{for all } t > s.$$

In particular $S(t)H_+ \subset H_+$.

Proof. By the variation of constants formula, $X_j(t) = e^{A_j(t)} X_j(s) + \int_s^t e^{A_j(t) - A_j(\theta)} k_{j-1} X_{j-1}^2(\theta)\,\mathrm{d}\theta$, where $A_j(t) = -\int_s^t k_j X_{j+1}(\theta)\,\mathrm{d}\theta$. □

We need also the following lemma which states that in H_+ the energy cannot increase. Given a solution X, denote by $E(t)$ the energy at time t, $E(t) = \sum_{j=1}^{\infty} X_j^2(t)$ and let $E_n(t) = \sum_{j=1}^{n} X_j^2(t)$.

Lemma 2.3. *Let X be a solution. If, for some $s \geqslant 0$, $X(s) \in H_+$, then $E(t) \leqslant E(s)$ for every $t \geqslant s$.*

Proof. By Lemma 2.2, if $X(s) \in H_+$, there is an n_0 such that $X_n(\theta) \geqslant 0$ for every $n \geqslant n_0$ and $\theta \in [s, t]$. Since $\frac{\mathrm{d}}{\mathrm{d}t}E_n = -2k_n X_n^2 X_{n+1}$, we deduce $\frac{\mathrm{d}}{\mathrm{d}t}E_{n-1}(\theta) \leqslant 0$ for all $\theta \in [s, t]$. Hence $E_{n-1}(t) \leqslant E_{n-1}(s)$, which implies, in the limit as $n \to \infty$, $E(t) \leqslant E(s)$. □

By Lemma 2.2, if $x \in H_+$ and X is a corresponding solution, then $X_n(t) > 0$ for all n larger than some n_0 and all $t \geqslant 0$, hence

$$a(t) \leqslant \sup_{n \leqslant n_0} k_n \left|X_n(t)\right| \leqslant k_{n_0} \sup_{n \leqslant n_0} \left|X_n(t)\right| \leqslant k_{n_0}\sqrt{E(t)} \leqslant k_{n_0}\sqrt{E(0)},$$

where the last inequality is due to Lemma 2.3. Hence solutions starting in H_+ are of class $\mathcal{K}$ and satisfy the weak energy inequality. Therefore:

Corollary 2.4. *If $x \in H_+$, system* (1) *has a unique global solution. In other words, $S(t)$ is univalued on H_+.*

3. On the strong energy inequality

Definition 3.1. A solution X on $[0,T)$ satisfies the *strong energy inequality* if

$$\|X(t)\|_H \leqslant \|X(s)\|_H \quad \forall s,t \in [0,T), \text{with } s<t. \tag{5}$$

We call *Leray–Hopf solution* a solution that satisfies the strong energy inequality.

For the Euler equations, the two notions of weak and strong energy inequalities are not equivalent, see [4]. Even for the Navier–Stokes equations (in 3D), it is only known the existence of solutions satisfying (2), and a weaker form of (5), namely for every $t>0$ and a.e. $s\in[0,t]$. In [4], for Euler equations, it is proved that even in the class of weak solutions satisfying (5), there are counterexamples to uniqueness. We complement the results of the previous section with the equivalence between (2) and (5), for the dyadic model.

Lemma 3.1. *If $E(t)<E(s)$ for some $t>s$, then $X(t)\in H_+$.*

Proof. By contradiction, if $X(t)\notin H_+$, there is a sequence $\{n_k\}_k$ such that for every k, $X_{n_k}(t)\leqslant 0$, yielding $X_{n_k}(\theta)\leqslant 0$ for all $\theta\in[s,t]$ (Lemma 2.2). Since $\frac{\mathrm{d}}{\mathrm{d}t}E_n=-2k_nX_n^2X_{n+1}$, we deduce $\frac{\mathrm{d}}{\mathrm{d}t}E_{n_k-1}(\theta)\geqslant 0$ for all $\theta\in[s,t]$, so that $E_{n_k-1}(t)\geqslant E_{n_k-1}(s)$. Since $E_n\uparrow E$, this implies $E(t)\geqslant E(s)$. □

Theorem 3.2. *A solution satisfies the weak energy inequality if and only if it satisfies the strong energy inequality.*

Proof. Let X be a solution satisfying (2) and let $0<s<t$. From (2) we have both $E(s)\leqslant E(0)$ and $E(t)\leqslant E(0)$. If $E(s)=E(0)$ we are finished. Otherwise $E(s)<E(0)$. So by Lemma 3.1, $X(s)\in H_+$, hence by Lemma 2.3, $E(s)\geqslant E(t)$. The proof is complete. □

Since we know existence of solutions with property (2), see [1], we immediately have:

Corollary 3.3. *For all initial conditions $x\in H$, there exists a global Leray–Hopf solution to system* (1).

Moreover, every solution with initial condition in H_+ is a Leray–Hopf solution. Whether every Leray–Hopf solution is of class $\mathcal{K}$ is an open question. We do not know counterexamples, since the example of non-uniqueness from [1] have increasing energy.

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