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Wilson spaces and homological algebra for coalgebraic modules

Les espaces de Wilson et l'algèbre homologique pour les modules cogèbriques

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ABSTRACT

In an earlier work, Wilson spaces were used to compute certain CTor Hopf algebras. In this Note we show how one can replace a resolution by infinite loop spaces associated to the Brown–Peterson spectrum with a resolution by Wilson spaces.

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RÉSUMÉ

Dans cet article, nous montrons que les espaces de Wilson peuvent être utilisés pour remplacer les espaces de lacets infinis associés au spectre de Brown–Peterson dans le calcul des CTor, les dérivées à gauche du produit tensoriel généralisé définies par Hunton et Turner.

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1. Introduction

In [13], Wilson studied the infinite loop spaces associated to the Brown–Peterson spectrum, and showed that both their homology and homotopy groups are torsion free. In [14], he further showed that they decompose as product of smaller spaces (see also [1,2]), which later became known as Wilson spaces (e.g. [12]).

On the other hand, the homology of infinite loop spaces tend to have rich structures, reflecting that of the cohomology theories these infinite loop spaces represent. Under certain conditions they become ring/module objects in the category of coalgebras, and it was shown by Hunton and Turner [6] that it was possible to carry out the homological algebra for such objects. Notably they defined the coalgebraic tensor product (see also [4] for an alternative approach) and its left derived functors CTor's.

Unfortunately there have been few results relating such homological algebra and homology of infinite loop spaces, due to the lack of computability of CTor's (e.g. [7,5] when only CTor_0 is involved, [9] when CTor_i's vanish for $i > 1$). One of the rare computations was made in [10] where we used Wilson spaces. More precisely, by definition, CTor's are homology (in the category of Hopf algebras) of certain chain complex (again, in the category of Hopf algebras) constructed using infinite loop spaces associated to the Brown–Peterson spectrum, and it was proved that this homology is isomorphic to the homology of another much smaller chain complex obtained using Wilson spaces [10, Propositions 8.1 and 8.4].

In this Note we formalize this process and prove Theorem 2.3, and show that such methods can be used efficiently to compute CTor's in certain cases.

Throughout the Note, p will be a fixed prime, BP will denote the Brown–Peterson spectrum at p (cf. [3]), so that $BP^{-*} = BP_* = Z_{(p)}[v_1, \ldots, v_n, \ldots]$ with $|v_n| = 2(p^n - 1)$, $BP\langle n\rangle$ denote the BP-module spectrum with $BP\langle n\rangle_* = Z_{(p)}[v_1, \ldots, v_n]$ with

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the convention $BP\langle 0\rangle = HZ_{(p)}$ (cf. [14,8]). To make arguments smooth, we commit an abuse of language and often identify a *BP*-module spectrum with its homotopy groups.

2. The machinery

First we fix terminology.

Definition 2.1. A Wilson space is a space that is a product of spaces of the form $\Omega^\infty \Sigma^k BP\langle n\rangle$ where $k \leqslant 2(1+p+\cdots+p^{n-1})$. We say that it is of finite type if its homotopy group is of finite type as a graded group.

Definition 2.2. Let M be a BP_*-module. A $\mathcal{W}$-resolution of M is an exact sequence of BP_*-modules $W. \to M$ such that

(i) $W.$'s are direct sums of $BP\langle n\rangle$'s.
(ii) $\Omega^\infty W.$'s are Wilson spaces of finite type.

If $\Sigma W.$ is still a $\mathcal{W}$-resolution of ΣM, then we say that it is a strong $\mathcal{W}$-resolution.

With this preparation, we are ready to state our result:

Theorem 2.3. *Let $h_*(-)$ be a generalized homology theory with Künneth isomorphism (e.g., h can be a Morava K-theory $K(n)$ or the ordinary homology $H_*(-, Z/p)$). Denote $k = h_*(pt)$. Let $W. \to M$ be a $\mathcal{W}$-resolution of a BP_*-module M. Suppose that M admits a free BP_*-resolution consisting of finite type modules. Then we have the following natural isomorphism*:

$$H_i\big(h_*\big(\Omega^\infty W.\big)\big) \cong \mathrm{CTor}_{i,0}^{k[BP^*]}\big(k[M], h_*(\underline{BP}_*)\big). \tag{1}$$

Proof. Without loss of generality, we can assume that both $W.$ and M are concentrated in even or odd degrees, thus we will do so without further notice. First we will treat the case of a strong $\mathcal{W}$-resolution. Suppose that $W. \to M$ is a strong $\mathcal{W}$-resolution of M, and $P. \to M$ a free BP_*-resolution of M. Then we get a chain map $P. \to W.$ compatible with the maps $P_0 \to M$ and $W_0 \to M$ at the module level, and these maps lift to maps of spectra. The associated maps of infinite loop spaces induces the chain map $h_*(\Omega^\infty P.) \to h_*(\Omega^\infty W.)$ after taking the $h_*(-)$. On the other hand, using [14, Theorem 5.4], one can construct the maps at infinite loop spaces' level that induce in $h_*(-)$ a chain homotopy inverse from $h_*(\Omega^\infty W.)$ to $h_*(\Omega^\infty P.)$ as in a standard proof of the uniqueness up to chain homotopy of projective resolutions (see, e.g., [11]).

Now suppose that $W. \to M$ is a $\mathcal{W}$-resolution which is not strong. In this case $W.$ and M are concentrated in even degrees, and $\Sigma^{-1}W. \to \Sigma^{-1}M$ is a strong $\mathcal{W}$-resolution of M. Take a free resolution $P.$ such that for each i the module P_i surjects to W_i. We still get a chain map $h_*(\Omega^\infty P.) \to h_*(\Omega^\infty W.)$ and we will show that it induces isomorphisms on homology. Since $h_*(\Omega^\infty \Omega P.)$, $h_*(\Omega^\infty \Omega W.)$ are exterior algebras [13,14] we can write

$$h_*\big(\Omega^\infty \Omega P.\big) \cong A. \otimes B.$$

where $A.$ is acyclic and $B. \cong h_*(\Omega^\infty \Omega W.)$ applying the above. It is also well known that the bar spectral sequence $\mathrm{Tor}^{h_*(\Omega^\infty \Omega X)}(h_*(pt), h_*(pt)) \Rightarrow h_*(\Omega^\infty X)$ collapses when $X = W.$ or $X = P.$, and we have $E_2 = E_\infty \cong \Gamma(\Sigma(Q(h_*(\Omega^\infty \Omega))))$ where Γ denotes the divided power coalgebra and Q the module of indecomposables (cf. [13,14]). Thus the E_∞ terms for $H_*(\Omega^\infty P.)$ are of the form $\Gamma\Sigma(Q(A.) \oplus Q(B.))$ whereas those for $H_*(\Omega^\infty W.)$ are of the form $\Gamma\Sigma(Q(B.))$. Since $Q(A.)$ is an acyclic complex of $h_*(pt)$-modules and Γ is exact, the desired result follows. □

3. Applications

Now we shall see how one can compute some CTor's using our theorem.

Corollary 3.1. *We have the following isomorphisms*:

$$\mathrm{CTor}_{i,2}^{Z/p[BP^*]}\big(Z/p\big[Z/\big(p^r\big)\big], H_*\big(\Omega^\infty \Sigma^* BP; Z/p\big)\big) \cong \begin{cases} H_*(CP^\infty; Z/p) & \textit{if } i = 0, 1, \\ Z/p & \textit{otherwise}, \end{cases} \tag{2}$$

$$\mathrm{CTor}_{i,1}^{Z/p[BP^*]}\big(Z/p\big[Z/\big(p^r\big)\big], H_*\big(\Omega^\infty \Sigma^* BP; Z/p\big)\big) \cong \begin{cases} H_*(S^1; Z/p) & \textit{if } i = 0, 1, \\ Z/p & \textit{otherwise}. \end{cases} \tag{3}$$

Proof. The sequence $\Sigma^j Z_{(p)} \to \Sigma^j Z_{(p)} \to \Sigma^j Z/(p^r)$ is a $\mathcal{W}$-resolution for $j = 1, 2$ noting that $Z_{(p)} = BP\langle 0\rangle$. The result follows. □

Remark 1. Let $P_\cdot \to M$ be a free *BP*-resolution of M. Denote Q_i the cofiber of $P_i \to P_{i-1}$. Thus we have a fibration sequence of infinite loop spaces

$$\cdots \to \Omega^\infty \Sigma^{l-1} Q_i \to \Omega^\infty \Sigma^l P_i \to \Omega^\infty \Sigma^l P_{i-1} \to \Omega^\infty \Sigma^l Q_i \to \cdots$$

which in turn gives rise to a sequence of Hopf algebras

$$\cdots \to h_*\big(\Omega^\infty \Sigma^{l-1} Q_i\big) \to h_*\big(\Omega^\infty \Sigma^l P_i\big) \to h_*\big(\Omega^\infty \Sigma^l P_{i-1}\big) \to h_*\big(\Omega^\infty \Sigma^l Q_i\big) \to \cdots.$$

If this sequence is long exact, then we get a spectral sequence

$$\mathrm{CTor}_{i,j}^{h_*[BP^*]}\big(h_*\big(\Omega^\infty \Sigma^* BP\big), h_*[M]\big) \Rightarrow h_*\big(\Omega^\infty \Sigma^{j-i} M\big).$$

Existence of such a spectral sequence should pave a way to systematic computations of homology of infinite loop spaces associated to *BP*-module spectra, which so far have been done by more or less ad-hoc method for each case.

In [9] it was shown that this actually is the case when CTor_i's vanish for all $i > 0$. However it doesn't happen for general M if $h = K(n)$ according to [10] (except if $n = 0$). From the corollary we see, for example

$$H_*\big(K(Z/p,1); Z/p\big) \cong \mathrm{CTor}_{1,2}^{Z/p[BP^*]}\big(Z/p[Z/p], H_*\big(\Omega^\infty \Sigma^* BP; Z/p\big)\big) \otimes \mathrm{CTor}_{0,1}^{Z/p[BP^*]}\big(Z/p[Z/p], H_*\big(\Omega^\infty \Sigma^* BP; Z/p\big)\big).$$

However $\mathrm{CTor}_{1,1}$ is non-trivial, which suggests that we don't have such a spectral sequence when $h = HZ/p$ either.

We can also apply our theorem to Wilson spaces themselves, considered as a resolution of length 1 to obtain the following:

Corollary 3.2. *Let M be a BP-spectrum such that $\Omega^\infty M$ is a Wilson space, and h be as in Theorem* 2.3. *Then the natural map*

$$\mathrm{CTor}_{0,0}^{h_*[BP^*]}\big(h_*\big(\Omega^\infty \Sigma^* BP\big), h_*[M]\big) \to h_*\big(\Omega^\infty M\big)$$

is an isomorphism.

Remark 2. It is also possible to prove Corollary 3.2 first and derive Theorem 2.3 from it. However, the passage from the case of strong $\mathcal{W}$-resolutions to the case of general $\mathcal{W}$-resolutions can be treated in a more natural way by considering entire resolutions than considering a single space, so we presented our results this way.

References

[1] J.M. Boardman, D.C. Johnson, W.S. Wilson, Unstable operations in generalized cohomology, in: Handbook of Algebraic Topology, North-Holland, Amsterdam, 1995, pp. 687–828.
[2] J.M. Boardman, W.S. Wilson, Unstable splittings related to Brown–Peterson cohomology, in: Cohomological Methods in Homotopy Theory, Bellaterra, 1998, in: Progr. Math., vol. 196, Birkhäuser, Basel, 2001, pp. 35–45.
[3] E.H. Brown, F.P. Peterson, A spectrum whose Z_p cohomology is the algebra of reduced p-th powers, Topology 5 (1966) 149–154.
[4] P.G. Goerss, Hopf rings, Dieudonné modules and $E_*(\Omega^2 S^3)$, in: J.-P. Meyer, J. Morava, W.S. Wilson (Eds.), Homotopy Invariant Algebraic Structures: A Conference in Honor of J. Michael Boardman, in: Contemp. Math., vol. 239, Amer. Math. Soc., Providence, RI, 1999, pp. 115–174.
[5] M.J. Hopkins, J.R. Hunton, On the structure of spaces representing a Landweber exact cohomology theory, Topology 34 (1995) 29–36.
[6] J.R. Hunton, P.R. Turner, Coalgebraic algebra, J. Pure Appl. Algebra 129 (1998) 297–313.
[7] J.R. Hunton, P.R. Turner, The homology of spaces representing exact pairs of homotopy functors, Topology 38 (1999) 621–623.
[8] D.C. Johnson, W.S. Wilson, *BP*-operations and Morava's extraordinary K-theories, Math. Z. 144 (1975) 55–75.
[9] T. Kashiwabara, Homological algebra for coalgebraic modules and mod p K-theory of infinite loop spaces, K-Theory 21 (2000) 387–417.
[10] T. Kashiwabara, W.S. Wilson, The Morava K-theory and Brown–Peterson cohomology of spaces related to *BP*, J. Math. Kyoto Univ. 41 (1) (2001) 43–95.
[11] S. MacLane, Homology, Die Grundlehren der mathematischen Wissenschaften, Bd. 114, Academic Press, Inc., Publishers, New York, Springer-Verlag, Berlin–Göttingen–Heidelberg, 1963.
[12] C.A. McGibbon, Wilson spaces and stable splittings of BT^r, Glasgow Math. J. 36 (3) (1994) 287–290.
[13] W.S. Wilson, The Ω-spectrum for Brown–Peterson cohomology, part I, Comment. Math. Helv. 4 (1973) 45–55.
[14] W.S. Wilson, The Ω-spectrum for Brown–Peterson cohomology, part II, Amer. J. Math. 97 (1975) 102–123.