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Geometry

An improved method for establishing Fuss' relations for bicentric polygons

Une méthode améliorée pour démontrer les relations de Fuss des polygones bicentriques

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ABSTRACT

This Note presents an improved method for proving Fuss' relations for bicentric n-gons where $n \geqslant 3$ is an odd integer. Several yet unknown Fuss type relations are established. The Note can be considered as a complement to one of our earlier articles on the same subject.

RÉSUMÉ

Ce travail présente une méthode améliorée pour démontrer les relations de Fuss pour des polygones bicentriques à n côtés, ou $n \geqslant 3$ est un nombre entier impair. Nous établissons des relations analogues à celles de Fuss, qui ne semblaient pas connues à ce jour. La note est un complément à un de nos articles antérieurs sur le même sujet.

1. Introduction

Although Poncelet's celebrated closure theorem [4] dates from the nineteenth century, many mathematicians have worked on a number of problems related to this inspiring result, which can be stated as follows. Let C and D be two nested conics such that there is an n-sided polygon inscribed in D and circumscribed around C. Then, for every point x on D there is an n-sided polygon inscribed in D and circumscribed around C such that the point x is one of its vertices. Hence, for every starting point x there is a polygon with the same n-periodicity.

In this article we restrict ourselves to the case when the conics are circles. The pair of conics C and D can be taken to be a pair of circles by a projective transformation. Let us denote by C_1 and C_2 the resulting circles, and let R, r and d be, respectively, the radius of C_2, the radius of C_1, and the distance between the centers of C_1 and C_2. The n-periodicity of Poncelet's configuration then implies algebraic relations on R, r and d. For $n \leqslant 8$, these relations were found by N. Fuss [2,3] and they are referred to as Fuss' relations for all values of n. A general condition on n-periodicity in terms of given conics is the content of the important Cayley's theorem [1] (which implies Fuss' relations; however the deduction of the later from the former may be a non-trivial task).

The present article primarily deals with one way of establishing Fuss' relations corresponding to the same value of n but different rotation numbers of Poncelet's n-gons. A key role is played in our argument by a certain partition of the rotation numbers for n, which allows one to relatively easily deduce Fuss' relations.

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2. One way of establishing Fuss' relations

The following notation will be used. We shall denote by

$$F_n^{(k)}(R,r,d)=0, \tag{1}$$

Fuss' relation for bicentric n-gons where the rotation number for n is k. Let (R_k, r_k, d_k) be a solution of the above relation. We then denote by $\hat{R}_k, \hat{r}_k, \hat{d}_k$ the lengths (which are, in fact, positive numbers) such that

$$(\hat{R}_k,\hat{r}_k,\hat{d}_k)=\left(\frac{R_k^2-d_k^2}{2r_k},\sqrt{-\left(R_k^2+d_k^2-r_k^2\right)+\left(\frac{R_k^2-d_k^2}{2r_k}\right)^2+\left(\frac{2R_kr_kd_k}{R_k^2-d_k^2}\right)^2},\frac{2R_kr_kd_k}{R_k^2-d_k^2}\right). \tag{2}$$

Let $n\geqslant 3$ be an odd integer and let us denote by $\mathbb{S}$ the set given by

$$\mathbb{S}=\left\{x\colon\ x\in\left\{1,2,\dots,\frac{n-1}{2}\right\}\text{ and } GCD(x,n)=1\right\}. \tag{3}$$

Definition 2.1. Let $f:\mathbb{S}\to\mathbb{S}$ be the function defined by

$$f(x)=2x\quad\text{if } 2x\in\mathbb{S},\quad\text{and}\quad f(x)=n-2x\quad\text{if } 2x\notin\mathbb{S}. \tag{4}$$

Theorem 2.2. *The function f is a one-to-one mapping from $\mathbb{S}$ to $\mathbb{S}$.*

Proof. It is easy to see that $x_1\neq x_2\Rightarrow f(x_1)\neq f(x_2)$. If $k\in\mathbb{S}$ is even, then the equation $2x=k$ has a solution in $\mathbb{S}$, whereas if k is odd, then the equation $k=n-2x$ has a solution in $\mathbb{S}$. □

Thus the function f induces a partition of the set $\mathbb{S}$.

For example, if $n=17$, then the partition of the set $\mathbb{S}=\{1,\dots,8\}$ has two cosets: $C_1=\{1,2,4,8\}$ and $C_2=\{3,5,6,7\}$, since in this case

$$f(1)=2,\qquad f(2)=4,\qquad f(4)=8,\qquad f(8)=1, \tag{5}$$

$$f(3)=6,\qquad f(6)=5,\qquad f(5)=7,\qquad f(7)=3. \tag{6}$$

Of course, the function f determines one (cyclic) ordering of the elements in each coset. For the sake of brevity, we shall write $x\to y$ instead of $f(x)=y$. Thus, if $n=17$, then instead of (5) and (6) we write the orderings $1\to2\to4\to8\to1$ and $3\to6\to5\to7\to3$.

Also, for brevity, we shall often write $\hat{x}$ instead of $f(x)$.

As will be seen, the ordering determined by the function f has very interesting and important properties concerning bicentric polygons. Namely, the following conjecture is strongly suggested:

Conjecture 2.3. *Let R_k, r_k, d_k and $\hat{R}_k, \hat{r}_k, \hat{d}_k$ be such that* (2) *holds. Then,*

$$(\hat{R}_k,\hat{r}_k,\hat{d}_k)=(R_{\hat{k}},r_{\hat{k}},d_{\hat{k}}),\quad\textit{that is,}\ \frac{R_k^2-d_k^2}{2r_k}=R_{f(k)},\ \textit{and so on}. \tag{7}$$

In [5, Theorems 1, 3, 4] we have proved this conjecture for $n=3,5,7,9$. So for $n=5$, since $\hat{1}=2$ and $\hat{2}=1$, we have the relations

$$(\hat{R}_1,\hat{r}_1,\hat{d}_1)=(R_2,r_2,d_2)\quad\text{and}\quad(\hat{R}_2,\hat{r}_2,\hat{d}_2)=(R_1,r_1,d_1). \tag{8}$$

We have also proved that

$$R_1\left(R_1-r_1+\sqrt{(R_1-r_1)^2-d_1^2}\right)=R_2^2, \tag{9}$$

$$R_2\left(R_2+r_2+\sqrt{(R_2+r_2)^2-d_2^2}\right)=R_1^2. \tag{10}$$

Generally, for each odd $n\geqslant 3$ for which Conjecture 2.3 is true, there are analogous relations

$$R_1\left(R_1-r_1+\sqrt{(R_1-r_1)^2-d_1^2}\right)=R^2_{\frac{n-1}{2}}, \tag{11}$$

$$R_2\left(R_2+r_2+\sqrt{(R_2+r_2)^2-d_2^2}\right)=R_1^2, \tag{12}$$

whose proof proceeds in the same way as that for $n=5,7,9$. Thus we have proved the following theorem:

Theorem 2.4. *Conjecture* 2.3 *is true for odd* $n = 3, 5, 7, 9, 11, 13, 15, 17$.

(*For odd* $n > 17$ *a powerful computer would be needed to ascertain the validity of Conjecture* 2.3.)

Now we shall show how, using relation (11), one can establish Fuss' relation for bicentric n-gons whose rotation numbers for n are odd integers from the set $\mathbb{S}$. Let this relation be denoted by $F_n^{\langle 1\rangle}(R, r, d) = 0$.

Without loss of generality we can take $n = 17$ since essentially the same argument applies in all of the other cases. First we shall use the coset $C_1 = \{1, 2, 4, 8\}$, where $1 \to 2 \to 4 \to 8 \to 1$. In this case the right-hand side of (11) is R_8^2, and thanks to (2) and (7) it can be expressed by R_1, r_1, d_1 using the following three substitutions:

$$(R_8, r_8, d_8) \leftarrow (R_4, r_4, d_4) \leftarrow (R_2, r_2, d_2) \leftarrow (R_1, r_1, d_1),$$

where the arrow $\leftarrow$ is read: can be expressed by.

It is clear from the ordering $1 \to 2 \to 4 \to 8 \to 1$ that (R_1, r_1, d_1) can be any solution of Fuss' relation $F_{17}^{(1)}(R, r, d) = 0$. Hence the relation thus obtained from (11), taking $n = 17$, is Fuss' relation $F_{17}^{(1)}(R, r, d) = 0$, except that we wrote R, r, d instead of R_1, r_1, d_1.

Now we use the coset $C_2 = \{3, 5, 6, 7\}$, where $3 \to 6 \to 5 \to 7 \to 3$. Since in this case $7 \to 3$, we have the following relation:

$$R_3\left(R_3 - r_3 + \sqrt{(R_3 - r_3)^2 - d_3^2}\right) = R_7^2. \tag{13}$$

The term R_7^2 can be expressed by R_3, r_3, d_3 using the following three substitutions:

$$(R_7, r_7, d_7) \leftarrow (R_5, r_5, d_5) \leftarrow (R_6, r_6, d_6) \leftarrow (R_3, r_3, d_3).$$

It is clear from the ordering $3 \to 6 \to 5 \to 7 \to 3$ that (R_3, r_3, d_3) can be any solution of Fuss' relation $F_{17}^{(3)}(R, r, d) = 0$. Hence the relation thus obtained from (13) is Fuss' relation $F_{17}^{(3)}(R, r, d) = 0$, except that we wrote R, r, d instead of R_3, r_3, d_3. In other words, Fuss' relation obtained for $8 \to 1$ is the same as Fuss' relation obtained for $7 \to 3$. In the same way, it can be seen that this also holds for $6 \to 5$ and $5 \to 7$. Hence the expression (relation) thus obtained is Fuss' relation for each of the rotation numbers 1, 3, 5, 7 for $n = 17$. In the same way, it can also be seen that it analogously holds for rotation numbers 2, 4, 6, 8 for $n = 17$. So, the relation (11) for $n = 17$ can be called a generator for Fuss' relation for bicentric 17-gons with odd rotation numbers for $n = 17$. Also, the relation (12) for $n = 17$ can be called the generator for Fuss' relation for bicentric 17-gons with even rotation numbers for $n = 17$.

We remark that in all examples considered we have found that the following holds. If m and n are odd integers such that each coset obtained for m has the same number of elements as each coset obtained for n, then we obtain an expression that is Fuss' relation for both m and n. So, for example, this is valid for $m = 7$ and $n = 9$, and for $m = 15$ and $n = 17$. Thus, relations (11) and (12) can be generators for Fuss' relations for bicentric n-gons with different odd n. (It seems that there are many other interesting properties, but one would require a powerful computer to investigate these.)

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