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Relaxed micro–macro schemes for kinetic equations

Schémas numériques micro–macro relaxés pour les équations cinétiques

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ABSTRACT

In Bennoune et al. (2008) [1], Lemou and Mieussens (2008) [6] we recently developed a general approach to design asymptotic preserving (AP) schemes for kinetic equations which are able to solve both macroscopic (small Knudsen number ε) and kinetic scales using the same model and the same numerical parameters. The strategy is based on micro/macro decompositions of the distribution function and can be applied to a large class of kinetic models (Boltzmann, Landau, etc.). However, the so-obtained schemes are shown to be AP for only close-to-equilibrium initial data in the non-linear case and, in general, they require costly inversions of non-local collision operators. In the present work, we introduce a new formulation of this strategy with the following properties: i) Initial data does not need to be well prepared, and may be independent of ε. ii) No inversion (even linear) of collision operators is needed and time-implicit schemes are obtained for the asymptotic models in the limit $\varepsilon \to 0$, making the schemes free from the usual diffusive CFL constraint. iii) The numerical schemes are consistent with the models for all values of $\varepsilon > 0$, and degenerate into consistent discretizations of the asymptotic model (Euler, Navier–Stokes, diffusion) when ε goes to 0, the numerical parameters (time–space–velocity steps) being fixed. Preliminary numerical validations of this approach are done on the non-local linear transport equation and its diffusion limit.

RÉSUMÉ

Dans Bennoune et al. (2008) [1], Lemou et Mieussens (2008) [6], nous avons récemment développé une approche générale pour construire des schémas numériques, dits AP (Asymptotic Preserving), capables de résoudre les équations cinétiques à différentes échelles (cinétique, fluide, diffusion). Cette stratégie est basée sur la décomposition micro–macro de la fonction distribution et peut s'appliquer à une large classe d'équations cinétiques (Boltzmann, Landau, etc.) et à différentes asymptotiques (Euler, Navier–Stokes, diffusion, etc.) quand le nombre de Knudsen ε tend vers 0. Cependant, elle nécessite généralement des inversions coûteuses d'opérateurs non locaux et la propriété (AP) n'est formellement vérifiée que pour des données initiales proches de l'équilibre dans le cas non linéaire. Dans ce travail, nous présentons une nouvelle formulation de cette stratégie ayant les propriétés suivantes : i) La donnée initiale est arbitraire et peut être choisie indépendante de ε. ii) Aucune inversion d'opérateur de collision n'est nécessaire, et un schéma implicite en temps est obtenu pour les modèles asymptotiques dans la limite $\varepsilon \to 0$, permettant de s'affranchir de la condition CFL diffusive usuelle. iii) Les schémas numériques obtenus sont consistants avec le modèle pour tout $\varepsilon > 0$, et dégénèrent en des discretizations consistantes avec les modèles asymptotiques (Euler, Navier–Stokes,

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diffusion) quand ε tend vers 0, les paramètres de discrétisation étant maintenus fixés. Pour valider l'approche, des tests numériques préliminaires sont éffectués dans le cas d'une équation de transport linéaire et non-local, et de sa limite de diffusion.

Version française abrégée

Le but de ce travail et de développer une nouvelle classe de schémas numériques multi-échelles pour les équations cinétiques. Il est en effet connu que si le nombre de Knudsen ε devient petit, une raideur apparaît dans ces modèles qui fait en sorte que leurs simulations numériques par des schémas explicites en temps deviennent rapidement inaccessibles (contraintes de type $\Delta t = O(\varepsilon)$ ou $O(\varepsilon^2)$). Alors qu'une stabilisation de ce type de raideur dans les schémas numériques correspondants pourrait se faire par des approches standards, la consistance de ces discrétisations avec les modèles asymptotiques quand $\varepsilon \to 0$ est en général une affaire délicate. Plusieurs travaux ont été éffectués pour construire des schémas dits AP (Asymptotic Preserving) pour des modèles cinétiques qui sont stables et consistants avec les limites de diffusion, voir par exemple [2–4]. D'autres références seront données et commentées dans la version detaillée [5].

Très récemment, nous avons introduit une approche générale permettant de construire des schémas numériques qui sont consistants avec les modèle cinétiques étudiés pour toutes les valeurs de $\varepsilon > 0$, stables dans la limite $\varepsilon \to 0$, et dégénèrent dans cette limite en des discrétisations consistantes avec les systèmes d'Euler, de Navier–Stokes compressibles [1], ou avec la limite de diffusion [6]. Un des avantages de cette approche est son adaptabilité aux diverses asymptotiques (fluide et diffusion par ex.). Elle est basée sur la décomposition micro–macro qui permet d'écrire de manière équivalente l'équation cinétique de départ sous la forme d'un système couplant une équation fluide ou de diffusion avec une équation sur la partie cinétique restante.

Cependant, dans le cas d'un opérateur de collision non linéaire, la propriété « AP » des approches [1,6] n'est formellement satisfaite que sous l'hypothèse d'une donnée initiale proche de l'équilibre. Par ailleurs, les schémas introduits dans [1,6] utilisent une semi-implicitation en temps de la raideur dans la partie cinétique, ce qui nécessite l'inversion d'opérateurs linéaires à chaque pas de temps et induit des coûts numériques importants (notamment en présence d'opérateurs de collision de type Boltzmann, Landau ou de type transport linéaire non local). De plus, par construction, la stratégie conduit à des schémas explicites en temps pour les modèles asymptotiques diffusifs. Dans ce travail, nous proposons de combiner cette approche avec des techniques de relaxation et construire ainsi des schémas numériques ayant les proprétés suivantes : i) La donnée initiale est arbitraire et peut être choisie loin de l'équilibre (et indépendante de ε). ii) Aucune inversion des opérateurs de collision (même linéarisés) n'est nécessaire, et les coefficients de diffusion dans les modèles limites sont précalculés. iii) Les schémas numériques obtenus sont consistants avec le modèle pour toutes les valeurs de ε, et dégénèrent en des discretizations consistantes avec les modèles asymptotiques (Euler, Navier–Stokes, diffusion) quand ε tend vers 0, les paramètres de discrétisation Δt, Δx, Δv, étant maintenus fixés. De plus une discrétisation adéquate de cette formulation conduit à un schéma implicite en temps pour les modèles diffusifs limites (NS, diffusion), ce qui permet de s'affranchir de la contrainte CFL diffusive usuelle $\Delta t = O(\Delta x^2)$. Cette Note est une version abrégée de [5].

1. Kinetic equations, their asymptotics and their micro/macro formulations

In order to design numerical schemes for kinetic equations which are able to solve both kinetic regime and macroscopic (fluid and diffusion) regime, a general class of numerical scheme has been introduced in [1,6] on the basis of micro/macro decompositions. Our aim was to develop a systematic way to construct numerical schemes which are consistent with the original model for all positive values of the Knudsen number ε and degenerates into consistent discretizations of the asymptotic models (Euler, NS, diffusion, etc.) when $\varepsilon \to 0$, the numerical parameters (Δt, Δx, Δv) being fixed (independent of ε) in this limit. Schemes with this property are also called 'Asymptotic Preserving' (AP). However, some requirements are needed for the schemes developed in [1,6] to be AP. First, the initial data should be close to equilibrium in the case of collision operators which are *non-linear* around the associated Maxwellian. Second, the linearized collision operator has to be inverted at each time iteration and the diffusion terms in the asymptotic schemes are explicit in time, suffering from the usual diffusive constraint. Our goal in this paper is to present a new formulation of the methodology that works for *arbitrary initial data*, *avoids the inversion* of the collision operators at each time step, and leads to *time-implicit schemes for the diffusive terms* in the limiting models, making the scheme free from the usual constraining diffusion CFL condition $\Delta t = O(\Delta x^2)$.

We first rewrite the micro/macro decomposition as in [1,6], but without using the 'near-equilibrium' assumption on the distribution functions. This decomposition will then be used in the next section to construct the desired schemes. This Note is an abridged version of [5].

1.1. Diffusion limit and kinetic/diffusion formulation of the linear transport equation

We first consider the following transport equation in a diffusive scaling

$$\varepsilon \partial_t f + v \cdot \nabla_x f = \frac{1}{\varepsilon} L f, \quad t > 0,\ (x, v) \in \mathbb{R}^d \times \mathbb{R}^d, \quad f|_{t=0} = f_{init} \tag{1}$$

where f is the distribution function of the particles that depends on time $t > 0$, on position $x \in \mathbb{R}^d$, and on velocity $v \in \Omega$. L is a linear operator which acts on the velocity dependence of f and describes the interactions of particles with the medium. We assume that there exits a positive equilibrium function $E = E(v)$ and that the collision operator L is non-positive self-adjoint in $L^2(\Omega, E^{-1}\,\mathrm{d}v)$; with nullspace and rank given by $\mathcal{N}(L) = \mathit{Span}\{E\} = \{f = \rho E,\ \text{where } \rho := \langle f\rangle\}$, $\mathcal{R}(L) = (\mathcal{N}(L))^{\perp} = \{f \text{ such that } \langle f\rangle := \int_\Omega f\,\mathrm{d}v = 0\}$. When ε goes to 0 in (1), it is easy to see that f converges to an equilibrium state $f_0 = \rho_0(t,x)E$. The diffusion limit is the equation satisfied by the density ρ_0 and is classically given by

$$\partial_t \rho_0 + \nabla_x \cdot (\kappa \nabla_x \rho_0) = 0, \quad \text{with } \kappa = \big\langle v L^{-1}(vE)\big\rangle. \tag{2}$$

Following [6], Eq. (1) can be written in a micro–macro equivalent form via the decomposition $f = \rho E + g$, with $\rho(t,x) = \langle f\rangle$ but with $g = f - \rho E$ not necessarily small as in [6]. Indeed, let Π be the orthogonal projector in $L^2(E^{-1}\,\mathrm{d}v)$ onto the nullspace of L: $\Pi\phi = \langle\phi\rangle E$. Then, injecting this decomposition into the kinetic equation and applying Π and $I - \Pi$ successively, one gets, after easy computations (see [6] for details),

$$\partial_t \rho + \frac{1}{\varepsilon}\nabla_x \cdot \langle vmg\rangle = 0, \qquad \partial_t g + \frac{1}{\varepsilon}(I - \Pi)(v \cdot \nabla_x g) = \frac{1}{\varepsilon^2}\big[Lg - \varepsilon(vE)\cdot\nabla_x \rho\big]. \tag{3}$$

1.2. Fluid asymptotics and kinetic/fluid formulation

We now consider non-linear kinetic equations in a fluid scaling

$$\partial_t f + v \cdot \nabla_x f = \frac{1}{\varepsilon} Q(f,f), \quad t > 0,\ (x,v) \in \mathbb{R}^d \times \mathbb{R}^d, \qquad f(t=0,x,v) = f_{init}(x,v), \tag{4}$$

where the collision operator Q is a quadratic operator acting only on the velocity dependence of the distribution function f. Fundamental examples are the well-known Boltzmann kernel for rarified gases and the Fokker–Planck–Landau operators for plasmas. In all what follows, we use the notations

$$m(v) = \big(1, v, |v|^2/2\big)^T, \quad \text{and} \quad \langle g\rangle = \int_{\mathbb{R}^d} g(v)\,\mathrm{d}v \tag{5}$$

for any scalar or vector function $g = g(v)$. It is well known that the Boltzmann and Landau operators $Q(f,f)$ have important physical properties of conservation and entropy: $\langle mQ(f,f)\rangle = 0$ and $\langle Q(f,f)\log(f)\rangle \leqslant 0$, $\forall f \geqslant 0$. It is also well known that the equilibrium functions (f such that $Q(f,f) = 0$) are Maxwellians:

$$M(U)(v) = \rho(2\pi T)^{-d/2}\exp\big(-|v-u|^2/2T\big), \quad U = \big(\rho, \rho u, \rho|u|^2/2 + d/2\big) = \big\langle mM(U)\big\rangle. \tag{6}$$

To any distribution function f, we shall associate its Maxwellian $M = M[f] = M(U)$, that is the function of the form (6) which has the same moments as f: $U = \langle mf\rangle = \langle mM[f]\rangle = \langle mM(U)\rangle$.

When ε goes to 0, f approaches a Maxwellian. At the first order in ε, the solution f approaches that of the compressible Euler system of gas dynamics. This is formally obtained by integrating the kinetic equation (4) against $m(v)$ and replacing f by its first order approximation: the Maxwellian which has the same first moments as f. At the second order in ε, a standard Chapman–Enskog expansion leads to the compressible Navier–Stokes system.

In order to design numerical schemes which are able to reproduce these asymptotics, a general class of numerical scheme has been introduced in [1] on the basis of micro/macro decompositions. Following [1] (with slight modifications), we decompose f as follows

$$f = M(U) + g, \tag{7}$$

where $U = \langle mf\rangle$ and $M(U)$ is the associated Maxwellian according to (6). When no confusion is possible, we set $M(U) = M$. Note that g is not a perturbation as in [1], and our goal here is to construct effective AP schemes, which do not require a well-prepared initial data. In other words, initial data may be chosen far from Maxwellians. We then insert decomposition (7) into (4) and respectively apply the operators Π_M and $I - \Pi_M$ (I is the identity operator and Π_M defined below). We then obtain without any approximation (see [1] for details), the following equivalent formulation of (4):

$$\begin{aligned} &\partial_t U + \nabla_x \cdot F(U) + \nabla_x \cdot \langle vmg\rangle = 0, \\ &\partial_t g + (I - \Pi_M)(v \cdot \nabla_x g) = \frac{1}{\varepsilon}\big[\mathcal{L}_M g + Q(g,g) - \varepsilon(I - \Pi_M)(v \cdot \nabla_x M)\big], \end{aligned} \tag{8}$$

where $F(U) = \langle vmM(U)\rangle$ is the usual Euler flux, $\mathcal{L}_M$ is the linearized collision operator given by $\mathcal{L}_M g = 2Q(M,g)$, and Π_M is the orthogonal projection in $L^2(M^{-1}\,\mathrm{d}v)$ onto $\mathcal{N}(\mathcal{L}_M) = \mathit{Span}\{M, vM, |v|^2 M\}$. Explicit expression of this projector is given in [1].

2. Relaxed micro/macro schemes (RMMS) for kinetic equation

We are now ready to present our numerical schemes. Note that only the time discretization will be investigated in this Note since space and velocity discretizations are essentially the same as those used in [1,6]. More detailed study and numerical tests will be given in [5]. As usual, we denote by f^n an approximation of a function f at time $t_n = n\Delta t$, $n \in \mathbb{N}$, $\Delta t > 0$.

2.1. Relaxed diffusion preserving schemes

We first recall the following time discretization of the micro–macro formulation (3), introduced in [6]:

$$\frac{\rho^{n+1}-\rho^n}{\Delta t} + \frac{1}{\varepsilon}\nabla_x \cdot \langle v g^{n+1}\rangle = 0, \qquad g^{n+1} = \left(I - \frac{\Delta t}{\varepsilon^2}L\right)^{-1}\left[g^n - \frac{\Delta t}{\varepsilon}(I-\Pi)\big(v\cdot\nabla_x g^n + vE\cdot\nabla_x\rho^n\big)\right]. \tag{9}$$

Due to the linearity of this model, it can be easily seen that this scheme is perfectly AP. However, it requires the inversion of a linear operator at each time step and, when $\varepsilon \to 0$, it degenerates into a time explicit scheme for the diffusion equation (2). Our aim is to apply relaxation procedures on the micro/macro formulations (3) in order to obtain an AP scheme which avoids the inversion of linear operator at each time step and degenerates into a time implicit scheme for the diffusion equation (2). Our relaxed AP scheme is given by

$$\begin{aligned}
&\frac{\rho^{n+1}-\rho^n}{\Delta t} + \big(1-e^{-\Delta t/\varepsilon^2}\big)\nabla_x\cdot\big[\kappa\nabla_x\rho^{n+1} + \big\langle v\cdot L^{-1}(I-\Pi)\big(v\cdot\nabla_x g^n\big)\big\rangle\big] + \frac{e^{-\Delta t/\varepsilon^2}}{\varepsilon}\nabla_x\cdot\langle v g^n\rangle = 0,\\
&g^{n+1} = e^{-\Delta t/\varepsilon^2}g^n + e^{-\Delta t/2\varepsilon^2}\big[Lg^n + g^n - \varepsilon(I-\Pi)\big(v\cdot\nabla_x g^n\big) - \varepsilon(vE)\cdot\nabla_x\rho^n\big].
\end{aligned} \tag{10}$$

To obtain this new formulation, we first write the second equation of (3) in two different Duhamel forms between $t_n = n\Delta t$ and $t_{n+1} = (n+1)\Delta t$ to get the two following approximations of g^{n+1}:

$$g^{n+1} = e^{-\Delta t/\varepsilon^2}g^n + \frac{1}{\varepsilon^2}\left(\int_{t_n}^{t_{n+1}} e^{-(t_{n+1}-t)/\varepsilon^2}\,\mathrm{d}t\right)\big[Lg^n + g^n - \varepsilon(I-\Pi)\big(v\cdot\nabla_x g^n\big) + (vE)\cdot\nabla_x\rho^n\big], \tag{11}$$

$$g^{n+1} = e^{\Delta t L/\varepsilon^2}g^n - \frac{1}{\varepsilon}\left(\int_{t_n}^{t_{n+1}} e^{(t_{n+1}-t)L/\varepsilon^2}\,\mathrm{d}t\right)(I-\Pi)\big(v\cdot\nabla_x g^n + (vE)\cdot\nabla_x\rho^n\big). \tag{12}$$

Approximating $\int_{t_n}^{t_{n+1}} e^{-(t_{n+1}-t)/\varepsilon^2}\,\mathrm{d}t$ by $\Delta t e^{-\Delta t/(2\varepsilon^2)}$ in (11) yields the second equation of scheme (10). We now briefly describe how we derived the first equation of (10) (see [5] for a detailed derivation). We use the approximation (12) of g^{n+1}, put it in the flux of the first equation in (9) and replace the integral $\int_{t_n}^{t_{n+1}} e^{(t_{n+1}-t)L/\varepsilon^2}\,\mathrm{d}t$ by its exact value $-\varepsilon^2(1 - e^{\Delta t L/\varepsilon^2})L^{-1}$. We now observe the following: For non-small ε, terms of order Δt can be removed in this flux and in particular $e^{\Delta t L/\varepsilon^2}$ may be replaced by $e^{-\Delta t/\varepsilon^2}$. For small ε ($\varepsilon \to 0$), both $e^{\Delta t L/\varepsilon^2}$ and $e^{-\Delta t/\varepsilon^2}$ go to 0. Therefore, $e^{\Delta t L/\varepsilon^2}$ may be replaced by $e^{-\Delta t/\varepsilon^2}$ in the so-obtained flux within all regimes (kinetic and diffusion). Finally, after making the diffusion term in ρ implicit in time, this yields the first equation of our relaxation scheme (10). The space discretization and boundary conditions are similar to that developed in [6] and a much more detailed study of boundary conditions and boundary layers in this context will be given in [5]. It is clear that scheme (10) satisfies the following: i) It degenerates into a consistent scheme for the diffusion equation on ρ, when ε goes to 0 and Δt is fixed. ii) If we fix $\varepsilon > 0$ and let Δt going to 0, then we see that this scheme is consistent with (3) and is first order in Δt. Finally, we note that the diffusion term in ρ of the first equation of (10) is implicit in time, which leads to a time-implicit scheme for the asymptotic diffusion model (2) when $\varepsilon \to 0$, and also makes scheme (10) free from the usual constrained diffusive CFL condition $\Delta t = O(\Delta x^2)$. This property cannot be obtained with the formulation (9) of [6]. Note that, when there is no boundary layer, the diffusion term in g in the first equation of scheme (10) can be neglected in both regimes. Indeed, for small ε, we deduce from the second equation of (10) that this term goes to 0 exponentially. For non-small ε (say $\varepsilon \sim 1$), this term is of the order of Δt and can then be neglected. Therefore, when there is no boundary layer, the inversion of the linear operator L is not needed to solve the original kinetic equation in both regimes. However, to treat boundary layers, it is necessary to keep this diffusion term and a special handling can be made at the boundary as in [6]. The numerical tests in [5] show that our scheme provides also good approximations of the boundary layers.

2.2. The relaxed Euler and Navier–Stokes preserving schemes

We first recall the time discretization of the micro–macro formulation (8), following [1]:

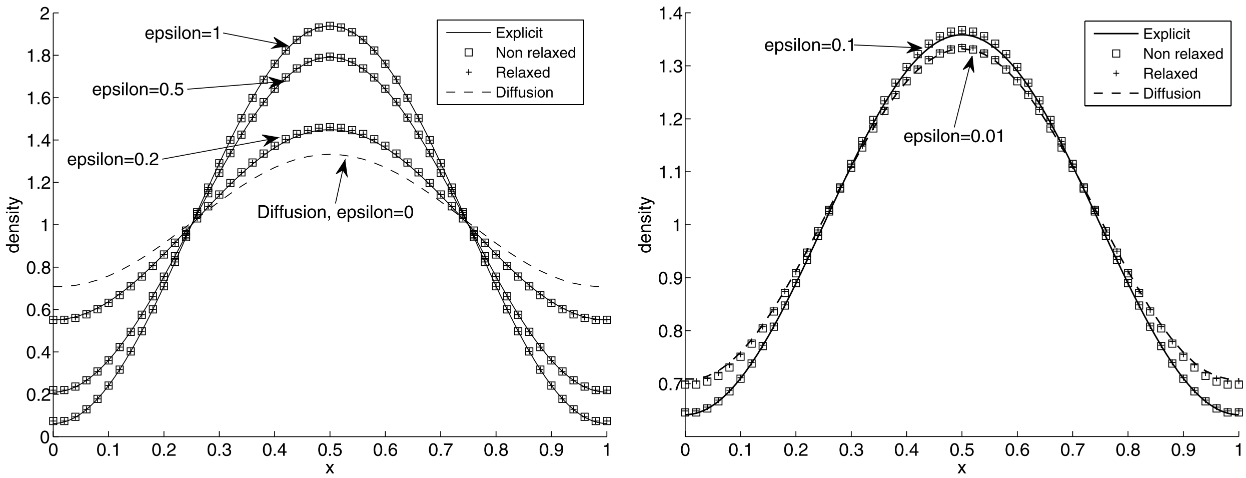

Fig. 1. *Kinetic and transition regimes* (left), $\varepsilon = 1, \varepsilon = 0.5, \varepsilon = 0.2$: density from the relaxed scheme (10) coincides with those obtained from the explicit scheme (in which $\Delta t = O(\varepsilon^2)$) and the non-relaxed micro/macro scheme (9). *Transition and diffusion regime* (right), $\varepsilon = 0.1$, $\varepsilon \leqslant 10^{-2}$: Relaxed micro–macro (10), non-relaxed micro/macro (9), and diffusion (2) coincide for $\varepsilon = 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}$, etc. Both figures are plotted at time $t = 0.1$. Relaxed and non-relaxed micro–macro schemes coincide with the reference solution obtained by a highly resolved explicit scheme ($\Delta t = O(\varepsilon^2)$) for all $\varepsilon \geqslant 10^{-2}$, and with that obtained by diffusion for $\varepsilon \leqslant 10^{-2}$.

$$\frac{U^{n+1} - U^n}{\Delta t} + \nabla_x \cdot F(U^n) + \nabla_x \cdot \langle vmg^{n+1} \rangle = 0, \qquad M^n = M(U^n),$$
$$g^{n+1} = \left(I - \frac{\Delta t}{\varepsilon} \mathcal{L}_{M^n}\right)^{-1} \left[g^n + \frac{\Delta t}{\varepsilon} Q(g^n, g^n) - \Delta t (I - \Pi_{M^n})(v \cdot \nabla_x g^n + v \cdot \nabla_x M^n)\right]. \tag{13}$$

In the case of collision operators which are linear around a Maxwellian (as the classical BGK), these schemes are perfectly AP since the stiff term $\frac{\Delta t}{\varepsilon} Q(g^n, g^n)$ does not exist in this case. However, in the non-linear case (as Boltzmann or Landau operators) these schemes are only AP under the assumption $g^n = O(\varepsilon)$, which is equivalent to taking close-to-equilibrium initial data. In addition, for all situations where a non-local collision operator is used (being linear or not), a costly linear inversion is needed at each time iteration to solve of (13) and the asymptotic scheme is a time explicit scheme for NS equations. To solve these problems, we proceed as for the above diffusion limit and get the following semi-discretization of system (8) (see [5] for details)

$$\frac{U^{n+1} - U^n}{\Delta t} + \nabla_x \cdot F(U^n) + \varepsilon(1 - e^{-\Delta t/\varepsilon}) \nabla_x \cdot \langle vm \mathcal{L}_{M^n}^{-1} (I - \Pi_{M^n})(v \cdot \nabla_x M^{n+1}) \rangle$$
$$+ \varepsilon(1 - e^{-\Delta t/\varepsilon}) \nabla_x \cdot \langle vm \mathcal{L}_{M^n}^{-1} (I - \Pi_{M^n})(v \cdot \nabla_x g^n) \rangle + e^{-\Delta t/\varepsilon} \nabla_x \cdot \langle vmg^n \rangle = 0,$$
$$g^{n+1} = e^{-\Delta t/\varepsilon} g^n + \frac{\Delta t}{\varepsilon} e^{-\Delta t/2\varepsilon} [Q(g^n, g^n) + \mathcal{L}_{M^n} g^n + g^n - \varepsilon(I - \Pi_{M^n})(v \cdot \nabla_x g^n + v \cdot \nabla_x M^n)]. \tag{14}$$

One can check formally that this scheme is AP in the following sense: i) It is consistent with (8) for all positive and fixed values of ε. ii) When $\varepsilon \to 0$, Δt being fixed, it degenerates into a consistent time discretization of the compressible Euler system up to the first order in ε, and of the compressible NS system up to the second order of ε, for arbitrary fixed initial data. The two fluxes in the first line of scheme (14) are nothing but the usual compressible Euler and NS fluxes. We also note that we have made implicit the diffusion term in the fluid part (the diffusion term in M^{n+1}, in the first equation of (14)), in order to obtain a time-implicit scheme for the asymptotic NS model. In this way, scheme (14) is free from the usual constrained diffusive CFL condition $\Delta t = O(\Delta x^2)$. This property cannot be obtained with the formulation (13) of [1]. Finally, as in the previous section, we observe that the diffusive term in g (in the first equation of (14)) can be neglected in all regimes when there is no boundary layer. Therefore, no inversion is needed to simulate the original kinetic equation in all regimes in this case, and the diffusion coefficients in the NS term (diffusion in M) should only be precalculated at the beginning of the simulation.

3. A numerical test

In this Note we propose to validate our strategy it in the case of a diffusion limit only. The other asymptotics will be simulated in a more detailed paper [5]. We present a simulation of Eq. (1) via our relaxed asymptotic preserving scheme given by (10). We consider the simple framework of one-dimensional spaces in $x \in [0, 1]$ and $v \in [-1, 1]$, and to shorten, we restrict ourselves to periodic boundary conditions in the space variable x. Dirichlet boundary conditions have been also tested and confirm that the scheme has all the AP properties described above. We take the following non-local form of L:

$$Lf(v) = \frac{1}{2}\int_{-1}^{1} \sigma(v, v_*)\big(f(v_*) - f(v)\big)\,\mathrm{d}v_*, \quad \sigma \geqslant 0, \quad \text{and} \quad \sigma(v, v_*) = \sigma(v_*, v), \quad \forall v, v_*.$$

For the numerical test, we choose: $\sigma(v, v_*) = |v - v_*|$. Finally, we take the initial data: $f^0(x, v) = 1 + \cos(2\pi(x + 1/2))$, whose micro–macro decomposition gives: $\rho^0(x) = 1 + \cos(2\pi(x + 1/2))$, $g^0 = 0$. In the following figures we plot the densities ρ obtained by the relaxed and non-relaxed micro–macro schemes given by (10) and (9), and compare them with the densities obtained from a direct explicit scheme for (1) and from the diffusion equation (2).

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