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Carleman estimates for degenerate parabolic equations with first order terms and applications

Une inégalitée de Carleman pour une équation parabolique dégenerée avec des termes de premier ordre et applications

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ABSTRACT

In this Note we prove a Carleman estimate for the one-dimensional degenerate parabolic equation

$$v_t + \left(x^\alpha v_x\right)_x - b(x,t)v + \left(x^{\beta/2} c(x,t)v\right)_x = 0, \quad (x,t) \in (0,1) \times (0,T),$$

where $\alpha \in [0,2)$, $\beta \geqslant \alpha$ and $b(x,t), c(x,t) \in L^\infty((0,1)\times(0,T))$ and give some controllability consequences.

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RÉSUMÉ

Dans cette Note on montre une inégalité de Carleman pour une équation unidimensionelle parabolique dégenerée

$$v_t + \left(x^\alpha v_x\right)_x - b(x,t)v + \left(x^{\beta/2} c(x,t)v\right)_x = 0, \quad (x,t) \in (0,1) \times (0,T),$$

où $\alpha \in [0,2)$, $\beta \geqslant \alpha$ et $b(x,t), c(x,t) \in L^\infty((0,1)\times(0,T))$ et on donne des conséquences en controllabilitée.

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Version française abrégée

Pour $T > 0$ fixe et un sous ensemble ouvert et non vide $\omega \subset (0,1)$ on considére le système linéaire

$$\begin{cases} y_t - (x^\alpha y_x)_x + b(x,t)y + x^{\beta/2}c(x,t)y_x = h1_\omega & \text{in } Q_1 = (0,1)\times(0,T), \\ y(1,t) = 0 \quad \text{and} \quad \begin{cases} y(0,t) = 0 & \text{if } 0 \leqslant \alpha < 1, \\ (x^\alpha y_x)(0,t) = 0 & \text{if } 1 \leqslant \alpha < 2, \end{cases} & t \in (0,T), \\ y(x,0) = y_0(x), & \text{in } (0,1). \end{cases} \tag{1}$$

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avec $b, c \in L^\infty(Q_1)$, $y_0 \in L^2(0,1)$, $\alpha \in [0,2)$ et $\beta \geqslant \alpha$. Ici, $h \in L^2(Q_1)$ est une fonction de contrôle (à déterminer), 1_ω est la fonction caractéristique de l'ensemble ω et y est la variable d'état. On dit que le système est contrôlable à zéro en temps $T > 0$ si, pour tout $y_0 \in L^2(0,1)$, il existe un contrôle $h \in L^2(Q_1)$ tel que $y(T) = 0$. Le résultat principal de cette Note est :

Théorème 0.1. *Soient $T > 0$ et $y_0 \in L^2(0,1)$, alors il existe $h \in L^2(Q_1)$ tel que la solution y de (1) satisfait*

$$y(T) = 0 \quad \textit{dans } [0,1].$$

De plus, il existe C une constante positive qui depend de T telle que

$$\int_0^T \int_\omega |h|^2 \, dx \, dt \leqslant C \int_0^1 y_0^2(x) \, dx.$$

La preuve du Théorème 0.1 s'appuie sur une inégalitée de Carleman pour le système

$$\begin{cases} v_t + (x^\alpha v_x)_x = F_0 + (x^{\beta/2} F_1)_x & \text{in } Q_1, \\ v(1,t) = 0 \quad \text{and} \quad \begin{cases} v(0,t) = 0 & \text{if } 0 \leqslant \alpha < 1, \\ (x^\alpha v_x)(0,t) = 0 & \text{if } 1 \leqslant \alpha < 2, \end{cases} & t \in (0,T), \\ v(x,T) = v_T(x), & \text{in } (0,1), \end{cases} \tag{2}$$

avec $F_0, F_1 \in L^2(Q_1)$ et $v_T \in L^2(0,1)$. Le résultat suivant est montré :

Lemme 0.1. *Soient $0 \leqslant \alpha < 2$ et $T > 0$ donnés. Alors il existe deux constantes positives C et s_0, telles que chaque solution v de (2) satisfait, pour tout $s \geqslant s_0$,*

$$\begin{aligned} &\int_0^T \int_0^1 \left(s\theta x^\alpha v_x^2 + s^3\theta^3 x^{2-\alpha} v^2\right) e^{2s\Phi(x,t)} \, dx \, dt \\ &\quad \leqslant C\left(\int_0^T \int_\omega e^{2s\Phi(x,t)} v^2 \, dx \, dt + \int_0^T \left(F_0^2 + s^2\theta^3 x^{\beta-\alpha} F_1^2\right) e^{2s\Phi(x,t)} \, dx \, dt\right) \end{aligned} \tag{3}$$

oú Φ et θ sont donnés dans (8) et (7).

1. Introduction and main result

Let us fix $T > 0$, a non-empty open subset $\omega \subset (0,1)$ and let us consider the linear system

$$\begin{cases} y_t - (x^\alpha y_x)_x + b(x,t)y + x^{\beta/2} c(x,t) y_x = h 1_\omega, & \text{in } Q_1 = (0,1) \times (0,T), \\ y(1,t) = 0 \quad \text{and} \quad \begin{cases} y(0,t) = 0 & \text{if } 0 \leqslant \alpha < 1, \\ (x^\alpha y_x)(0,t) = 0 & \text{if } 1 \leqslant \alpha < 2, \end{cases} & t \in (0,T), \\ y(x,0) = y_0(x), & \text{in } (0,1). \end{cases} \tag{4}$$

where $b, c \in L^\infty(Q_1)$, $y_0 \in L^2(0,1)$, $\alpha \in [0,2)$ and $\beta \geqslant \alpha$. Here, $h \in L^2(Q_1)$ is a control function (to be determined), 1_ω the characteristic function of the set ω and y is the state variable. It will be said that (4) is *null controllable* at time T if, for each $y_0 \in L^2(0,1)$, there exists a control $h \in L^2(Q_1)$ such that

$$y(T) = 0 \quad \text{in } [0,1]. \tag{5}$$

Let us recall that the above problem is well-posed in appropriate weighted spaces. For $0 \leqslant \alpha < 1$, define the Hilbert space $H_\alpha^1(0,1)$ as

$$H_\alpha^1(0,1) := \left\{u \in L^2(0,1) \;\middle|\; u \text{ absolutely continuous in } [0,1], x^{\alpha/2} u_x \in L^2(0,1) \text{ and } u(0) = u(1) = 0\right\},$$

and the unbounded operator $A : D(A) \subset L^2(0,1) \to L^2(0,1)$ by

$$\forall u \in D(A), \quad Au := \left(x^\alpha u_x\right)_x, \qquad D(A) := \left\{u \in H_\alpha^1(0,1) \;\middle|\; x^\alpha u_x \in H^1(0,1)\right\}.$$

For $1 \leqslant \alpha < 2$, let us change the definition of $H_\alpha^1(0,1)$ to

$$H_\alpha^1(0,1) := \left\{u \in L^2(0,1) \;\middle|\; u \text{ locally absolutely continuous in } (0,1], \; x^{\alpha/2} u_x \in L^2(0,1) \text{ and } u(1) = 0\right\}.$$

Then, the unbounded operator $A : D(A) \subset L^2(0,1) \to L^2(0,1)$ will be defined by

$$\begin{cases} \forall u \in D(A), \quad Au := (x^\alpha u_x)_x, \\ D(A) := \big\{ u \in L^2(0,1) \;\big|\; u \text{ locally absolutely continuous in } (0,1], \\ \qquad\qquad x^\alpha u \in H_0^1(0,1),\ x^\alpha u_x \in H^1(0,1) \text{ and } (x^\alpha u_x)(0) = 0 \big\}. \end{cases}$$

In both cases $0 \leqslant \alpha < 1$ and $1 \leqslant \alpha < 2$, the following results hold (see, e.g., [2]):

Proposition 1.1. *$A : D(A) \subset L^2(0,1) \to L^2(0,1)$ is a closed self-adjoint negative operator with dense domain.*

Hence, A is the infinitesimal generator of a strongly continuous semigroup e^{tA} on $L^2(0,1)$. Consequently, we have the following well-posedness result:

Theorem 1.2. *Let h be given in $L^2(Q_1)$. For all $y_0 \in L^2(0,1)$, (4) has a unique solution*

$$y \in \mathcal{U} := \mathcal{C}^0\big([0,T]; L^2(0,1)\big) \cap L^2\big(0,T; H_\alpha^1(0,1)\big).$$

Moreover, if $y_0 \in D(A)$, then

$$y \in \mathcal{C}^0\big([0,T]; H_\alpha^1(0,1)\big) \cap L^2\big(0,T; D(A)\big) \cap H^1\big(0,T; L^2(0,1)\big).$$

The controllability properties of linear one-dimensional degenerate parabolic equations with zero order terms are nowadays well known; see for instance [1,6,12,7]. In [5] the authors study the property of null controllability of the system (4) with first order terms when the coefficient c only depends on spatial variable x. In this Note we are considering coefficients of the first order terms of the form $x^{\beta/2}c(x,t)$ with $\beta \geqslant \alpha$ and $c(x,t) \in L^\infty((0,1)\times(0,T))$. This dependence on t improves the null controllability results existing in the literature and will allow to prove a null controllability result for a semilinear equation for certain non-linearities $f = f(x,t,y,y_x)$. This issue will be presented in an extended version of this Note [9]. We also extend the results in [3,4] where the *regional* null controllability of equations of the form of (4) is considered. Our main result is the following one:

Theorem 1.3. *Given $T > 0$ and $y_0 \in L^2(0,1)$, there exists $h \in L^2(Q_1)$ such that the solution y of (4) satisfies*

$$y(T) = 0 \quad \textit{in } [0,1].$$

Moreover, for some positive constant C that depends on T,

$$\int_0^T \int_\omega |h|^2 \,\mathrm{d}x\,\mathrm{d}t \leqslant C \int_0^1 y_0^2(x)\,\mathrm{d}x.$$

2. Carleman inequality

The proof of our main result is a consequence of this crucial estimate of Carleman type, that will be useful to prove an observability inequality for the adjoint system. Let us consider the parabolic system

$$\begin{cases} v_t + (x^\alpha v_x)_x = F_0 + (x^{\beta/2}F_1)_x, & \text{in } Q_1, \\ v(1,t) = 0 \quad \text{and} \quad \begin{cases} v(0,t) = 0 & \text{if } 0 \leqslant \alpha < 1, \\ (x^\alpha v_x)(0,t) = 0 & \text{if } 1 \leqslant \alpha < 2, \end{cases} & t \in (0,T), \\ v(x,T) = v_T(x), & \text{in } (0,1), \end{cases} \tag{6}$$

where $F_0, F_1 \in L^2(Q_1)$ and $v_T \in L^2(0,1)$. Observe that, for every $F_0, F_1 \in L^2(Q_1)$ and $v_T \in L^2(0,1)$, (6) admits a unique weak solution (see [11]) that satisfies $v \in \mathcal{C}^0([0,T]; L^2(0,1)) \cap L^2(0,T; H_\alpha^1(0,1))$ and $v_t \in L^2(0,T; (H_\alpha^1)'(0,1))$.

For $\omega = (a,b)$ let us call $\kappa = \frac{2a+b}{3}$, $\lambda = \frac{a+2b}{3}$ and $\xi \in C^2(\mathbb{R})$ be such that $0 \leqslant \xi \leqslant 1$ and

$$\xi(x) = \begin{cases} 1, & \text{if } x \in (0,\kappa), \\ 0, & \text{if } x \in (\lambda,1). \end{cases}$$

Let us define

$$\theta(t) = \frac{1}{(t(T-t))^4}\ \forall t \in (0,T), \qquad \psi(x) = \begin{cases} (x^{2-\alpha} - c_1), & 0 \leqslant \alpha < 2,\ \alpha \neq 1,\ \forall x \in [0,1], \\ (e^x - c_1), & \alpha = 1,\ \forall x \in [0,1] \end{cases} \tag{7}$$

where c_1 is such that $\psi(x) < 0$ for every $x \in [0,1]$. Now, let us set

$$\zeta(x)=\frac{1-x^{\alpha/2}}{1-\alpha/2},\qquad \Psi(x)=e^{r\zeta(x)}-e^{2r\zeta(0)}$$

and the negative function

$$\Phi(x,t)=\theta(t)\big[\xi(x)\psi(x)+(1-\xi(x))\Psi(x)\big]. \tag{8}$$

The main result of this section is the following:

Lemma 2.1. *Let $0\leqslant\alpha<2$ and $T>0$ be given. Then, there exist two positive constants C and s_0, such that every solution v of (6) satisfies, for all $s\geqslant s_0$,*

$$\int_0^T\int_0^1\big(s\theta x^\alpha v_x^2+s^3\theta^3x^{2-\alpha}v^2\big)e^{2s\Phi(x,t)}\,\mathrm{d}x\,\mathrm{d}t\leqslant C\left(\int_0^T\int_\omega e^{2s\Phi(x,t)}v^2\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\big(F_0^2+s^2\theta^3x^{\beta-\alpha}F_1^2\big)e^{2s\Phi(x,t)}\,\mathrm{d}x\,\mathrm{d}t\right). \tag{9}$$

Proof. We will prove Lemma 2.1 following the technique developed in [10]; that is to say, we will use recent Carleman inequalities, such as those developed in [7], for degenerate parabolic equations with second member zero to obtain the wished inequality. The proof is done in two steps:

Step 1. Two auxiliary null control problems

In [8], is proved that for system (6) with right-hand side zero; there exist two positive constants C and s_0, such that for all $s\geqslant s_0$,

$$\int_0^T\int_0^1\big(s\theta x^\alpha v_x^2+s^3\theta^3x^{2-\alpha}v^2\big)e^{2s\Phi(x,t)}\,\mathrm{d}x\,\mathrm{d}t\leqslant C\int_0^T\int_\omega e^{2s\Phi(x,t)}v^2\,\mathrm{d}x\,\mathrm{d}t. \tag{10}$$

Let $s\geqslant s_0$ and let us introduce the null controllability problem: Given $f\in L^2(Q_1)$, find $u\in L^2(Q_1)$ such that the solution $z\in L^2(0,T;H^1_\alpha(0,1))$ to

$$\begin{cases} z_t-(x^\alpha z_x)_x=s^3\theta^3x^{2-\alpha}e^{2s\Phi(x,t)}f+u1_\omega, & \text{in } Q_1,\\ z(1,t)=0 \quad\text{and}\quad \begin{cases} z(0,t)=0 & \text{if } 0\leqslant\alpha<1,\\ (x^\alpha z_x)(0,t)=0 & \text{if } 1\leqslant\alpha<2,\end{cases} & t\in(0,T),\\ z(x,0)=0, & \text{in } (0,1),\end{cases}$$

satisfies $z(x,T)=0$ in $(0,1)$ and

$$\int_0^T\int_0^1 e^{-2s\Phi(x,t)}z^2\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\int_\omega e^{-2s\Phi(x,t)}u^2\,\mathrm{d}x\,\mathrm{d}t\leqslant C\int_0^T\int_0^1 s^3\theta^3x^{2-\alpha}e^{2s\Phi(x,t)}f^2\,\mathrm{d}x\,\mathrm{d}t. \tag{11}$$

Following [10] it is possible to prove that inequality (10) implies the existence of such a control and that inequality (11) is satisfied. We apply the previous considerations to the solution $v\in L^2(Q_1)$ to (6) and deduce the existence of a control $\hat u\in L^2(Q_1)$ and state $\hat z\in L^2(0,T;H^1_\alpha(0,1))$ such that

$$\begin{cases} \hat z_t-(x^\alpha \hat z_x)_x=s^3\theta^3x^{2-\alpha}e^{2s\Phi(x,t)}v+\hat u1_\omega, & \text{in } Q_1,\\ \hat z(1,t)=0 \quad\text{and}\quad \begin{cases} \hat z(0,t)=0 & \text{if } 0\leqslant\alpha<1,\\ (x^\alpha \hat z_x)(0,t)=0 & \text{if } 1\leqslant\alpha<2,\end{cases} & t\in(0,T),\\ \hat z(x,0)=\hat z(x,T)=0, & \text{in } (0,1),\end{cases} \tag{12}$$

and

$$\int_0^T\int_0^1 e^{-2s\Phi(x,t)}\hat z^2\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\int_\omega e^{-2s\Phi(x,t)}\hat u^2\,\mathrm{d}x\,\mathrm{d}t\leqslant C\int_0^T\int_0^1 s^3\theta^3x^{2-\alpha}e^{2s\Phi(x,t)}v^2\,\mathrm{d}x\,\mathrm{d}t. \tag{13}$$

Now, if we multiply by $s^{-2}\theta^{-3}e^{-2s\Phi(x,t)}\hat z$ the equation satisfied by $\hat z$ and we integrate by parts, we infer

$$\int_0^T\int_0^1 s^{-2}\theta^{-3}x^\alpha e^{-2s\Phi(x,t)}\hat z_x^2\,\mathrm{d}x\,\mathrm{d}t\leqslant C\int_0^T\int_0^1 s^3\theta^3x^{2-\alpha}e^{2s\Phi(x,t)}v^2\,\mathrm{d}x\,\mathrm{d}t,$$

that together with inequality (13) gives

$$\int_0^T\int_0^1 e^{-2s\Phi(x,t)}\hat{z}^2\,dx\,dt+\int_0^T\int_\omega e^{-2s\Phi(x,t)}\hat{u}^2\,dx\,dt+\int_0^T\int_0^1 s^{-2}\theta^{-3}x^\alpha e^{-2s\Phi(x,t)}\hat{z}_x^2\,dx\,dt$$
$$\leqslant C\int_0^T\int_0^1 s^3\theta^3x^{2-\alpha}e^{2s\Phi(x,t)}v^2\,dx\,dt \tag{14}$$

for $s\geqslant s_0$.

Similarly, for $x^{\alpha/2}v_x\in L^2(Q_1)$, where v is solution to (6); we can deduce the existence of a control $\tilde{u}\in L^2(Q_1)$ and state $\tilde{z}\in L^2(0,T;H^1_\alpha(0,1))$ such that

$$\begin{cases}\tilde{z}_t-(x^\alpha\tilde{z}_x)_x=s\theta(e^{2s\Phi(x,t)}x^{\alpha/2}v_x)_x+\tilde{u}1_\omega, & \text{in } Q_1,\\ \tilde{z}(1,t)=0 \quad\text{and}\quad \begin{cases}\tilde{z}(0,t)=0 & \text{if } 0\leqslant\alpha<1,\\ (x^\alpha\tilde{z}_x)(0,t)=0 & \text{if } 1\leqslant\alpha<2,\end{cases} & t\in(0,T),\\ \tilde{z}(x,0)=\tilde{z}(x,T)=0, & \text{in } (0,1),\end{cases} \tag{15}$$

and

$$\int_0^T\int_0^1 e^{-2s\Phi(x,t)}\tilde{z}^2\,dx\,dt+\int_0^T\int_\omega e^{-2s\Phi(x,t)}\tilde{u}^2\,dx\,dt+\int_0^T\int_0^1 s^{-2}\theta^{-2}x^\alpha e^{-2s\Phi(x,t)}\tilde{z}_x^2\,dx\,dt$$
$$\leqslant C\int_0^T\int_0^1 s\theta x^\alpha e^{2s\Phi(x,t)}v_x^2\,dx\,dt \tag{16}$$

for $s\geqslant s_0$.

Step 2. Proof of inequality (9)
If we multiply by v Eq. (12) and we integrate by parts, we have

$$\int_0^T\int_0^1 s^3\theta^3x^{2-\alpha}e^{2s\Phi(x,t)}v^2\,dx\,dt$$
$$\leqslant\left(\int_0^T\int_0^1 e^{2s\Phi(x,t)}F_0^2\,dx\,dt+\int_0^T\int_0^1 s^2\theta^3x^{\beta-\alpha}e^{2s\Phi(x,t)}F_1^2\,dx\,dt+\int_0^T\int_\omega e^{2s\Phi(x,t)}v^2\,dx\,dt\right)^{1/2}$$
$$\times\left(\int_0^T\int_0^1 e^{-2s\Phi(x,t)}\hat{z}^2\,dx\,dt+\int_0^T\int_0^1 s^{-2}\theta^{-3}x^\alpha e^{-2s\Phi(x,t)}\hat{z}_x^2\,dx\,dt+\int_0^T\int_\omega e^{-2s\Phi(x,t)}v^2\,dx\,dt\right)^{1/2}.$$

Now, taking into account (14), we deduce

$$\int_0^T\int_0^1 s^3\theta^3x^{2-\alpha}e^{2s\Phi(x,t)}v^2\,dx\,dt\leqslant C\Bigg(\int_0^T\int_0^1 e^{2s\Phi(x,t)}F_0^2\,dx\,dt+\int_0^T\int_0^1 s^2\theta^3x^{\beta-\alpha}e^{2s\Phi(x,t)}F_1^2\,dx\,dt$$
$$+\int_0^T\int_\omega e^{2s\Phi(x,t)}v^2\,dx\,dt\Bigg). \tag{17}$$

Similarly, if we multiply by v Eq. (15), we integrate by parts and we apply (16); we can conclude

$$\int_0^T\int_0^1 s\theta x^\alpha e^{2s\Phi(x,t)}v_x^2\,dx\,dt\leqslant C\left(\int_0^T\int_0^1 e^{2s\Phi(x,t)}\left[F_0^2+s^2\theta^3x^{\beta-\alpha}F_1^2\right]dx\,dt+\int_0^T\int_\omega e^{2s\Phi(x,t)}v^2\,dx\,dt\right),$$

that, together with (17), gives (9). The theorem is now proved. □

3. Null controllability of system (4). Proof of Theorem 1.3

The key point in the proof of Theorem 1.3 is to obtain an appropriate observability inequality for the corresponding adjoint equation to (4). That is, consider v solution to

$$\begin{cases} v_t + (x^\alpha v_x)_x - b(x,t)v + (x^{\beta/2}c(x,t)v)_x = 0, & \text{in } Q_1, \\ v(1,t) = 0 \quad \text{and} \quad \begin{cases} v(0,t) = 0 & \text{if } 0 \leqslant \alpha < 1, \\ (x^\alpha v_x)(0,t) = 0 & \text{if } 1 \leqslant \alpha < 2, \end{cases} & t \in (0,T), \\ v(x,T) = v_T(x), & \text{in } (0,1), \end{cases} \tag{18}$$

with $v_T \in L^2(0,1)$. It is now well understood that null controllability of (4) is equivalent to the following result:

Lemma 3.1. *Let $T > 0$ be given. Then there exists a positive constant C such that every solution v of* (18) *satisfies*

$$\int_0^1 v^2(x,0)\,dx \leqslant C \int_0^T \int_\omega v^2(x,t)\,dx\,dt. \tag{19}$$

The proof of (19) combines Carleman inequality (9) with energy estimates with the Hardy–Poincaré type inequality [13]:

$$\int_0^1 x^{\alpha-2} u^2\,dx \leqslant \frac{4}{(1-\alpha)^2} \int_0^1 x^\alpha {u_x}^2\,dx,$$

where u is a locally absolutely continuous function in $(0,1)$ and

$$u(x) \to_{x\to 0^+} 0, \quad \text{if } 0 \leqslant \alpha < 1; \qquad u(x) \to_{x\to 1^-} 0,$$

if $1 < \alpha < 2$ and, in both cases,

$$\int_0^1 x^\alpha {u_x}^2\,dx < \infty.$$

Finally, using the observability inequality (19) and a standard technique, one can prove Theorem 1.3.

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References

[1] F. Alabau-Boussouira, P. Cannarsa, G. Fragnelli, Carleman estimates for degenerate parabolic operators with applications to null controllability, J. Evol. Equ. 6 (2) (2006) 161–204.
[2] M. Campiti, G. Metafune, D. Pallara, Degenerate self-adjoint evolution equations on the unit interval, Semigroup Forum 57 (1998) 1–36.
[3] P. Cannarsa, G. Fragnelli, Null controllability of semilinear degenerate parabolic equations in bounded domains, EJDE 136 (2006) 1–20.
[4] P. Cannarsa, G. Fragnelli, J. Vancostenoble, Regional controllability of semilinear degenerate parabolic equations in bounded domains, J. Math. Anal. Appl. 320 (2) (2006) 804–818.
[5] P. Cannarsa, G. Fragnelli, D. Rocchetti, Null controllability of degenerate parabolic operators with drift, Netw. Heterog. Media 2 (4) (2007) 695–715 (electronic).
[6] P. Cannarsa, P. Martinez, J. Vancostenoble, Persistent regional null controllability for a class of degenerate parabolic equations, Commun. Pure Appl. Anal. 3 (2004) 607–635.
[7] P. Cannarsa, P. Martinez, J. Vancostenoble, Carleman estimates for degenerate parabolic operators, SIAM J. Control Optim. 47 (1) (2008) 1–19.
[8] P. Cannarsa, L. de Teresa, Insensitizing controls for one-dimensional degenerate parabolic equations, EJDE 2009 (73) (2009) 1–21.
[9] C. Flores, L. de Teresa, Null controllability of one-dimensional degenerate parabolic equations with first order terms, in preparation.
[10] O.Yu. Imanuvilov, M. Yamamoto, Carleman inequalities for parabolic equations in Sobolev spaces of negative order and exact controllability for semilinear parabolic equations, Publ. Res. Inst. Math. Sci. 39 (2) (2003) 227–274.
[11] J.-L. Lions, E. Magenes, Non-Homogeneous Boundary Value Problems and Applications, vol. I, Die Grundlehren der mathematischen Wissenschaften, Band 181, Springer-Verlag, New York/Heidelberg, 1972.
[12] P. Martinez, J. Vancostenoble, Carleman estimates for one-dimensional degenerate heat equations, J. Evol. Equ. 6 (2) (2006) 325–362.
[13] B. Opic, A. Kufner, Hardy-Type Inequalities, Longman Scientific and Technical, Harlow, UK, 1990.