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# Sharp inequalities related to Gosper's formula

## *Inégalités précises liées à la formule de Gosper*

Cristinel Mortici

*Valahia University of Târgovişte, Department of Mathematics, Bd. Unirii 18, 130082 Târgovişte, Romania*

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A B S T R A C T

The purpose of this Note is to construct a new type of Stirling series, which extends the Gosper's formula for big factorials. New sharp inequalities for the gamma and digamma functions are established. Finally, numerical computations which demonstrate the superiority of our new series over the classical Stirling's series are given.



R É S U M É

Le but de cette Note est de construire un nouveau type de série de Stirling, étendant la formule de Gosper pour les grandes factorielles. Nous établissons de nouvelles inégalités précises pour les fonctions gamma et digamma. Enfin, nous indiquons des calculs numériques qui démontre la supériorité de notre nouvelle série sur la série classique de Stirling.



Probably one from the most known approximation formula for the factorial function is the Stirling formula:

$$n! \approx \sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^n,$$

that is the first estimate of the Stirling series, *e.g.*, [1, p. 257],

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} - \frac{517}{2488320n^4} + \cdots\right). \tag{1}$$

In this Note we discuss the following more accurate formula:

$$n! \approx \sqrt{2\pi\left(n + \frac{1}{6}\right)} \cdot \left(\frac{n}{e}\right)^n \quad (\text{Gosper [4]}),$$

proving the following double inequality for $x \geqslant 1$,

$$\sqrt{2\pi\left(x + \frac{1}{6}\right)}\left(\frac{x}{e}\right)^x < \Gamma(x+1) \leqslant \omega \cdot \sqrt{2\pi\left(x + \frac{1}{6}\right)}\left(\frac{x}{e}\right)^x, \tag{2}$$

*E-mail address:* cmortici@valahia.ro.

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where the constant $\omega = e\sqrt{\frac{3}{7\pi}} = 1.0039940821\ldots$ is the best possible. Furthermore, we prove new sharp bounds for the digamma function $\psi$, *e.g.* [1, p. 258]. For $x \geqslant 1$, it holds

$$\ln x - \frac{1}{x} + \frac{1}{2(x+\frac{1}{6})} - \zeta \leqslant \psi(x) < \ln x - \frac{1}{x} + \frac{1}{2(x+\frac{1}{6})}, \tag{3}$$

with the best possible constant $\zeta = \gamma - \frac{4}{7} = 0.0057864\ldots$ ($\gamma = 0.577215\ldots$ is the Euler–Mascheroni constant), improving other known results [2,5–7,14] as

$$\ln x - \frac{1}{x} < \psi(x) < \ln x - \frac{1}{2x}, \quad x > 1.$$

Instead of completing the approximation series at the end part of the formula, as in the classical Stirling's series, we continue the series under the square root, the first two terms being given by the Gosper's formula. In this way, we obtain a series converging unexpectedly faster than the well-known Stirling series, namely

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi\left(n + \frac{1}{6} + \frac{1}{72n} - \frac{31}{6480n^2} - \frac{139}{155520n^3} + \frac{9871}{6531840n^4} + \cdots\right)},$$

which can be viewed as an extension of the Stirling and Gosper formulas.

## 2. The results

In the first part of this section, we prove the following:

**Theorem 1.** *The function $f : [1,\infty) \to \mathbb{R}$ given by*

$$f(x) = \ln\Gamma(x+1) - x\ln x + x - \ln\sqrt{2\pi} - \frac{1}{2}\ln\left(x + \frac{1}{6}\right)$$

*is convex, and strictly decreasing.*

**Proof.** We have

$$f'(x) = \psi(x) + \frac{1}{x} - \ln x - \frac{1}{2(x+\frac{1}{6})}, \qquad f''(x) = \psi'(x) - \frac{1}{x^2} - \frac{1}{x} + \frac{1}{2(x+\frac{1}{6})^2}.$$

According to a result of Gordon [3, Theorem 4], for every $x > 0$, we have

$$\psi'(x) > \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5}.$$

As for every $x \geqslant 1$,

$$\left(\frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5}\right) - \left(\frac{1}{x^2} + \frac{1}{x} - \frac{1}{2(x+\frac{1}{6})^2}\right) = \frac{45 - \frac{31}{x} - \frac{12}{x^2} - \frac{1}{x^3}}{30x^8(6x+1)^2} > 0,$$

it results that $f''(x) > 0$, for every $x \geqslant 1$. Now, $f'$ is strictly increasing with $\lim_{x\to\infty} f'(x) = 0$, thus $f' < 0$ and $f$ is strictly decreasing. $\square$

As a direct consequence of the fact that $f$ is strictly decreasing, we have $0 = \lim_{x\to\infty} f(x) < f(x) \leqslant f(1) = 1 + \ln\sqrt{\frac{3}{7\pi}}$, for every $x \geqslant 1$, which is the sharp inequality (2).

Using the fact that $f'$ is strictly increasing, we have $\frac{4}{7} - \gamma = f'(1) \leqslant f'(x) < \lim_{x\to\infty} f'(x) = 0$, for every $x \geqslant 1$, which is the sharp inequality (3).

Now we are interested in finding the real values $a, b, c, d$ that provide the most accurate formula of the form

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi\left(n + \frac{1}{6} + \frac{a}{n} + \frac{b}{n^2} + \frac{c}{n^3} + \frac{d}{n^4}\right)}. \tag{4}$$

To this end, we introduce the sequence $(\lambda_n)_{n\geqslant 1}$ by the relations

$$n! = \left(\frac{n}{e}\right)^n \sqrt{2\pi\left(n + \frac{1}{6} + \frac{a}{n} + \frac{b}{n^2} + \frac{c}{n^3} + \frac{d}{n^4}\right)} \cdot \exp\lambda_n, \tag{5}$$

and we say that an approximation (5) is better, the faster $(\lambda_n)_{n\geqslant 1}$ converges to zero. A basic tool for measuring the rate of convergence is the following:

**Lemma.** *If* $(\lambda_n)_{n\geqslant 1}$ *is convergent to zero and there exists the limit*

$$\lim_{n\to\infty} n^k(\lambda_n - \lambda_{n+1}) = l \in \overline{\mathbb{R}}, \tag{6}$$

*with* $k > 1$, *then there exists the limit*: $\lim_{n\to\infty} n^{k-1}\lambda_n = \frac{l}{k-1}$.

This lemma was used by Mortici [8–13] for constructing asymptotic expansions, or accelerating some convergences. For proof, see, *e.g.*, [11].

We can see from the lemma that the speed of convergence of the sequence $(\lambda_n)_{n\geqslant 1}$ is even higher as the value $k$ satisfying (6) is greater.

As we are interested to compute a limit of the form (6), we write $\lambda_n - \lambda_{n+1}$ from (5) as a power series of $n^{-1}$,

$$\begin{aligned}\lambda_n - \lambda_{n+1} &= \left(\frac{1}{72} - a\right)\frac{1}{n^3} + \left(\frac{7a}{4} - \frac{3b}{2} - \frac{17}{540}\right)\frac{1}{n^4} + \left(a^2 - \frac{23a}{9} + \frac{10b}{3} - 2c + \frac{641}{12960}\right)\frac{1}{n^5} \\ &\quad + \left(\frac{1505}{432}a - \frac{425}{72}b + \frac{65}{12}c - \frac{5}{2}d + \frac{5}{2}ab - \frac{35}{12}a^2 - \frac{1831}{27216}\right)\frac{1}{n^6} \\ &\quad + \left(\frac{169}{18}b - \frac{493}{108}a - \frac{34}{3}c + 8d - \frac{17}{2}ab + 3ac + \frac{51}{8}a^2 - a^3 + \frac{3}{2}b^2 + \frac{55609}{653184}\right)\frac{1}{n^7} + O\left(\frac{1}{n^8}\right).\end{aligned}$$

The fastest sequence $(\lambda_n)_{n\geqslant 1}$ is obtained when the first four coefficients of this power series vanish. In this case

$$a = \frac{1}{72}, \qquad b = \frac{-31}{6480}, \qquad c = \frac{-139}{155\,520}, \qquad d = \frac{9871}{6531840},$$

we have

$$\lambda_n - \lambda_{n+1} = \frac{324179}{391910400n^7} + O\left(\frac{1}{n^8}\right)$$

and by the lemma, the sequence $(\lambda_n)_{n\geqslant 1}$, representing the expression of the error term, converges with asymptotically relative error $\frac{324179}{2351\,462\,400n^6}$.

Finally, we demonstrate the superiority of our new series

$$n! \approx \sqrt{2\pi\left(n + \frac{1}{6} + \frac{1}{72n} - \frac{31}{6480n^2} - \frac{139}{155\,520n^3} + \frac{9871}{6531840n^4}\right)}\left(\frac{n}{e}\right)^n = \mu_n \tag{7}$$

over the classical Stirling series truncated at the $n^{-4}$ term (see Table 1):

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} - \frac{517}{2488320n^4}\right) = \sigma_n. \tag{8}$$

**Table 1**
Numerical results.

| $n$ | $\sigma_n - n!$ | $n! - \mu_n$ |
|---|---|---|
| 50 | $2.9137 \times 10^{52}$ | $2.4477 \times 10^{50}$ |
| 100 | $1.2919 \times 10^{145}$ | $1.2301 \times 10^{142}$ |
| 500 | $3.9298 \times 10^{1118}$ | $1.0671 \times 10^{1114}$ |
| 2500 | $8.9183 \times 10^{7392}$ | $8.6741 \times 10^{7386}$ |

In fact, the Stirling series (8) remains weaker than our formula (7), even if we omit from (7) the term containing $n^{-4}$.

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