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Mathematical Analysis

Self-similar sets with initial cubic patterns

Ensembles auto-similaires avec motifs initiaux cubiques

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ABSTRACT

For $A \subset \{0, \dots, n-1\}^m$, let E_A be the unique nonempty compact subset of $\mathbb{R}^m$ such that $E_A = \bigcup_{a \in A}(\frac{1}{n}E_A + \frac{a}{n})$. We show that two such self-similar sets E_A and E_B (for $A, B \subset \{0, \dots, n-1\}^m$), supposed to be totally disconnected, are Lipschitz equivalent if and only if $\#A = \#B$.

RÉSUMÉ

Si $A \subset \{0, \dots, n-1\}^m$, soit E_A l'unique compact non vide de $\mathbb{R}^m$ tel que $E_A = \bigcup_{a \in A}(\frac{1}{n}E_A + \frac{a}{n})$. Nous montrons que deux tels ensembles auto-similaires totalement discontinus E_A et E_B (avec $A, B \subset \{0, \dots, n-1\}^m$) sont lipschitziennement équivalents si et seulement si $\#A = \#B$.

Version française abrégée

Deux espaces métriques (X_1, d_1) et (X_2, d_2) sont dits *lipschitziennement équivalents*, et l'on écrira alors $X_1 \simeq X_2$, s'il existe une bijection $f : X_1 \to X_2$ et une constante $C > 0$ telles que pour tous $u, v \in X_1$, $C^{-1} d_1(u, v) \leqslant d_2(f(u), f(v)) \leqslant C d_1(u, v)$.

Etant donné un entier $n \geqslant 2$ et un ensemble $A \subset \{0, \dots, n-1\}^m$, nous dirons que

$$E_A = \bigcup_{a \in A}\left(\frac{1}{n}E_A + \frac{a}{n}\right) \tag{1}$$

est l'ensemble auto-similaire déterminé par le motif initial $\{\frac{1}{n}[0,1]^m + \frac{a}{n}\}_{a \in A}$.

Nous étudions ici l'équivalence lipschitzienne des ensembles auto-similaires du type précédent. L'équivalence lipschitzienne d'ensembles auto-similaires a été intensément étudiée (par exemple [1–4,6–9]). En particulier, pour $m = 1$, $n = 5$ et $A_1 = \{0, 2, 4\}$, $A_2 = \{0, 3, 4\}$, il a été démontré [6] que $E_{A_1} \simeq E_{A_2}$, ce qui répond à une question de David et Semmes [2, Problème 11.16]. Le théorème suivant généralise le résultat de [6], qui se place dans le cas $m = 1$:

Théorème 0.1. *Soit $A, B \subset \{0, 1, \dots, n-1\}^m$ tels que les ensembles auto-similaires correspondants E_A et E_B soient totalement discontinus. Alors $E_A \simeq E_B$ si et seulement si $\#A = \#B$.*

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De plus, nous avons :

Théorème 0.2. *Soit $C_1 \subset \{0, 1, \ldots, (n_1 - 1)\}^{m_1}$, $C_2 \subset \{0, 1, \ldots, (n_2 - 1)\}^{m_2}$ et les ensembles auto-similaires $F_{C_1} = \bigcup_{c_1 \in C_1} (\frac{1}{n_1} F_{C_1} + \frac{c_1}{n_1})(\subset \mathbb{R}^{m_1})$ et $F_{C_2} = \bigcup_{c_2 \in C_2} (\frac{1}{n_2} F_{C_1} + \frac{c_2}{n_2})(\subset \mathbb{R}^{m_2})$ supposés tous deux totalement discontinus. Alors $F_{C_1} \simeq F_{C_2}$ si et seulement s'il existe $k_1, k_2 \in \mathbb{N}$ tels que $n_1^{k_1} = n_2^{k_2}$ et $(\#C_1)^{k_1} = (\#C_2)^{k_2}$.*

Si $a \in A \subset \mathbb{R}^m$, on pose $S_a(x) = (x + a)/n$. On pose aussi $S_{a_1 \cdots a_k} = S_{a_1} \circ S_{a_2} \circ \cdots \circ S_{a_k}$. Pour tout entier $t \geqslant 1$, posons $\Psi_t = \bigcup_{a_1 \cdots a_t \in A^t} (S_{a_1 \cdots a_t}[0, 1]^m)$.

Définition 0.3. *On dit que E_A est de type fini, s'il existe un entier M tel que pour tout entier k, chaque composante connexe de Ψ_k contient au plus M cubes de taille n^{-k}.*

Avec ces notions, nous avons le théorème suivant dont le Théorème 0.1 est un cas particulier :

Théorème 0.4. *Soit $A \subsetneq \{0, 1, \ldots, n - 1\}^m$ et $E_A = \bigcup_{a \in A} (\frac{1}{n} E_A + \frac{a}{n}) \subset \mathbb{R}^m$ un ensemble auto-similaire. Alors les conditions suivantes sont équivalentes :*

(i) *E_A est totalement discontinu ;*
(ii) *E_A est de type fini ;*
(iii) *E_A et $\Sigma_{n,\#A}$ sont lipschitziennement équivalents, où $\Sigma_{n,\#A}$ désigne le système symbolique $\{1, \ldots, \#A\}^\infty$ muni de la métrique $d_{n,\#A}(x_1 x_2 \cdots, y_1 y_2 \cdots) = n^{-\min\{k | x_k \neq y_k\}}$.*

1. Introduction

Two metric spaces (X_1, d_1) and (X_2, d_2) are said to be *Lipschitz equivalent*, denoted by $X_1 \simeq X_2$, if there exists a bijection $f : X_1 \to X_2$ and a constant $C > 0$ such that for all $u, v \in X_1$, $C^{-1} d_1(u, v) \leqslant d_2(f(u), f(v)) \leqslant C d_1(u, v)$.

Given an integer $n \geqslant 2$ and a set $A \subset \{0, \ldots, n - 1\}^m$, the unique compact subset of $\mathbb{R}^m$ such that

$$E_A = \bigcup_{a \in A} \left(\frac{1}{n} E_A + \frac{a}{n} \right) \tag{2}$$

will be called, self-similar set with initial *cubic pattern* $\{\frac{1}{n}[0, 1]^m + \frac{a}{n}\}_{a \in A}$.

Here, we study the Lipschitz equivalence of self-similar sets with initial cubic patterns. The Lipschitz equivalence between self-similar sets has been extensively studied ([1–4,6–9]...). In particular, for $m = 1$, $n = 5$, and $A_1 = \{0, 2, 4\}$, $A_2 = \{0, 3, 4\}$, it is proved [6] that $E_{A_1} \simeq E_{A_2}$, which answers a question [2, Problem 11.16] by David and Semmes about the Lipschitz equivalence of self-similar sets with different "patterns" (or "rules" in Sections 2.5, 11.7 and Chapter 13 of [2]). The following Theorem 1.1 extends the result of [6], which takes place in $\mathbb{R}$:

Theorem 1.1. *Let A and B be subsets of $\{0, 1, \ldots, n - 1\}^m$ such that the corresponding self-similar sets E_A and E_B are totally disconnected. Then $E_A \simeq E_B$ if and only if $\#A = \#B$.*

With a little more effort, we can get

Theorem 1.2. *Let $C_1 \subset \{0, 1, \ldots, (n_1 - 1)\}^{m_1}$, $C_2 \subset \{0, 1, \ldots, (n_2 - 1)\}^{m_2}$ and suppose that both*

$$F_{C_1} = \bigcup_{c_1 \in C_1} \left(\frac{1}{n_1} F_{C_1} + \frac{c_1}{n_1} \right) (\subset \mathbb{R}^{m_1}) \quad \text{and} \quad F_{C_2} = \bigcup_{c_2 \in C_2} \left(\frac{1}{n_2} F_{C_1} + \frac{c_2}{n_2} \right) (\subset \mathbb{R}^{m_2})$$

are totally disconnected. Then $F_{C_1} \simeq F_{C_2}$ if and only if there are $k_1, k_2 \in \mathbb{N}$ such that $n_1^{k_1} = n_2^{k_2}$ and $(\#C_1)^{k_1} = (\#C_2)^{k_2}$.

2. Preliminaries

2.1. Symbolic system and graph-directed sets

Let $\Sigma_{n,l}$ be the set $\{1, \ldots, l\}^\infty$ $(n, l \geqslant 2)$ with the metric $d_{n,l}(x_1 x_2 \cdots, y_1 y_2 \cdots) = n^{-\min\{k | x_k \neq y_k\}}$. The next lemma follows from [1,4] or Proposition 11.8 of [2]:

Lemma 2.1. *$(\Sigma_{n_1,l_1}, d_{n_1,l_1}) \simeq (\Sigma_{n_2,l_2}, d_{n_2,l_2})$ if and only if there are $k_1, k_2 \in \mathbb{N}$ such that $n_1^{k_1} = n_2^{k_2}$ and $l_1^{k_1} = l_2^{k_2}$.*

We recall the notion "graph directed sets" [5]. Let (X,d) be a compact metric space and $G=(V,\Gamma)$ a directed graph such that, for each edge $e\in\Gamma$, there is a corresponding similarity $T_e:(X,d)\to(X,d)$ with ratio $r_e<1$ (i.e., $d(T_ex_1,T_ex_2)=r_ed(x_1,x_2)$ for all $x_1,x_2\in X$). Assume that for each vertex $j\in V$, there is at least one edge starting from j. Then there is a unique family $\{K_i\}_{i\in V}$ of nonempty compact subsets of X such that for any $i\in V$,

$$K_i=\bigcup_{j\in V}\bigcup_{e\in\mathcal{E}_{i,j}}T_e(K_j),\tag{3}$$

where $\mathcal{E}_{i,j}$ is the set of edges linking i to j. If (3) is a disjoint union for each $i\in V$, we say that $\{K_i\}_{i\in V}$ are dust-like graph-directed sets on (V,Γ). Theorem 2.1 of [6] yields the following lemma:

Lemma 2.2. *Suppose $\{K_i\}_{i\in V}$ and $\{K_i'\}_{i\in V}$ are dust-like graph-directed sets on (V,Γ) satisfying (3) and $K_i'=\bigcup_{j\in V}\bigcup_{e\in\mathcal{E}_{i,j}}T_e'(K_j')$, where $T_e':(X',d')\to(X',d')$ for some compact metric space (X',d'). If for each $e\in\Gamma$ the corresponding similarities T_e and T_e' have the same ratio ρ_e, then $K_i\simeq K_i'$ for each $i\in V$.*

2.2. Connectedness

For $a\in A\subset\mathbb{R}^m$, set $S_a(x)=(x+a)/n$. Write $S_{a_1\cdots a_k}=S_{a_1}\circ S_{a_2}\circ\cdots\circ S_{a_k}$. The Hausdorff distance between two subsets E and f of $\mathbb{R}^m$ is defined to be $d_H(E,F)=\max[\sup_{x\in E}d(x,F),\sup_{y\in F}d(E,y)]$.

Due to compactness, we get the following lemmas:

Lemma 2.3. *Suppose $\{X_k\}_k$ are connected compact subsets of $[-r,r]^m$ $(r>0)$. Then there exist a subsequence $\{k_i\}_i$ and a connected compact set X such that $X_{k_i}\overset{d_H}{\to}X$ as $i\to\infty$.*

Lemma 2.4. *Let $\{Y_i\}_{i=1}^t$ be totally disconnected compact subsets of $\mathbb{R}^m$. Then $Y=\bigcup_{i=1}^tY_i$ is totally disconnected.*

For every subset $X\subset\mathbb{R}^m$, let $int(X)$ and ∂X denote its interior and boundary respectively.

Considering the unit cube $[0,1]^m$ and its neighbors of the same size, we have

$$[-1,2]^m=\bigcup_{h\in\{-1,0,1\}^m}\big([0,1]^m+h\big).$$

Set

$$\Xi_k=\bigcup_{h\in\{-1,0,1\}^m}\bigg(\bigcup_{a_1a_2\cdots a_k\in A^k}S_{a_1\cdots a_k}\big([0,1]^m\big)+h\bigg).\tag{4}$$

Lemma 2.5. *Suppose that E_A is totally disconnected. Then there exists an integer k such that for any connected component χ of Ξ_k with $\chi\cap[0,1]^m\neq\emptyset$, χ is contained in $(-1,2)^m$.*

Proof. Suppose on the contrary that for any k there are connected components $\chi_k\subset\Xi_k$ and points $x_k\in[0,1]^m\cap\chi_k$ and $y_k\in\partial[-1,2]^m\cap\chi_k$ with $h\in\{-1,0,1\}^m$. By Lemma 2.3, we can take a subsequence $\{k_i\}_i$ such that $x_{k_i}\to x^*\in[0,1]^m$, $y_{k_i}\to y^*\in\partial[-1,2]^m$, and $\chi_{k_i}\overset{d_H}{\to}\Gamma$ for some *connected* and compact set Γ with $\Gamma\subset\bigcup_{h\in\{-1,0,1\}^m}(E_A+h)$, $x^*\in\Gamma\cap[0,1]^m$, and $y^*\in\Gamma\cap\partial[-1,2]^m$. However $\bigcup_{h\in\{-1,0,1\}^m}(E_A+h)$ is totally disconnected by Lemma 2.4. This is a contradiction. □

2.3. Finite type

For any integer $t\geqslant1$, set $\Psi_t=\bigcup_{a_1\cdots a_t\in A^t}(S_{a_1\cdots a_t}[0,1]^m)$.

Definition 2.6. We say E_A is of finite type if there is an integer M such that for every integer k, any connected component of Ψ_k contains at most M cubes of side n^{-k}.

Remark 1. Here is an equivalent definition of the finite type property: E_A is of finite type, if there are positive integers M_0 and k_0 such that for every integer k, any connected component of Ψ_{kk_0} contains at most M_0 cubes of side n^{-kk_0}.

In this subsection, we always assume that E_A is of finite type. Fix an integer k^* large enough so that

$$(\#A)^{k^*}>M^2\quad\text{and}\quad(\sqrt{m}M)n^{-k^*}<1/3.\tag{5}$$

Then for any connected component χ of Ψ_{k^*}, $diam(\chi) \leqslant (\sqrt{m}M)n^{-k^*} < 1/3$. For any vertex $z = (z_1, \ldots, z_m) \in \{0,1\}^m$ of $[0,1]^m$, set $\Lambda_z = \{(y_1, \ldots, y_m)\colon\ y_i = 1 - z_i$ for some $i\}$ and

$$\Delta_{z,k^*} = \{\chi\colon\ \chi \text{ is a connected component of } \Psi_{k^*} \text{ and } \chi \cap \Lambda_z = \emptyset\}.$$

For any connected component χ of Ψ_{k^*} and any $i \in \mathbb{N} \cap [1,m]$, since $diam(\chi) < 1/3$, the set χ intersects at most one of the following sets $\{(y_1, \ldots, y_m)\colon\ y_i = 1\}$ and $\{(y_1, \ldots, y_m)\colon\ y_i = 0\}$, i.e., $\chi \in \Delta_{z,k^*}$ for some $z \in \{0,1\}^m$. That means the set $\bigcup_{z\in\{0,1\}^m} \Delta_{z,k^*}$ consists of all connected components of Ψ_{k^*}. Taking subset $\bar{\Delta}_{z,k^*}$ of Δ_{z,k^*}, such that

$$\Psi_{k^*} = \bigcup_{z\in\{0,1\}^m} \bigcup_{\chi\in\bar{\Delta}_{z,k^*}} \chi \quad \text{and} \quad \bar{\Delta}_{z,k^*} \cap \bar{\Delta}_{z',k^*} = \emptyset \tag{6}$$

for $z \neq z' \in \{0,1\}^m$. Here Δ_{z,k^*} and $\bar{\Delta}_{z,k^*}$ may be empty.

We call $D \subset \mathbb{Z}^m$ a *type*, if $\bigcup_{d\in D}([0,1]^m + d)$ is connected. We say D_1 and D_2 are equivalent, denoted by $D_1 \sim D_2$, if there exists $z \in \mathbb{Z}^m$ such that $D_1 = D_2 + z$. We say that one type D_3 is generated by type D_4, if there exist $z \in \mathbb{Z}^m$ and an integer $l \geqslant 1$ such that $\bigcup_{d\in D_3}([0,1]^m + d)$ is a connected component of $\bigcup_{d\in D_4}[\Psi_{lk^*} + d]/n^{-lk^*} + z$.

If E_A is of finite type, then there are finitely many types $\{D_1, \ldots, D_{N(A)}\}$, with $N(A) \leqslant 2^{[(2M)^m]}$, generated by type $D_0 = \{\mathbf{0}\}$ and such that $D_i \nsim D_j$ for any $i \neq j$. Given $D_i \in \{D_0, D_1, \ldots, D_{N(A)}\}$ and $z \in \{0,1\}^m$, we fix a point $d_{z,i} \in D_i$ such that $d_{z,i} + z$ is an *extreme vertex* of $\bigcup_{d\in D_i}([0,1]^m + d)$. Here the phrase *extreme vertex* means that if $d_{z,i} + z \in [0,1]^m + d'$ with $d' \in D_i$, then $d' = d_{z,i}$. Set

$$G_i = \bigcup_{z\in\{0,1\}^m} \left(\bigcup_{\chi\in\bar{\Delta}_{z,k^*}} \chi + d_{z,i} \right) \quad \text{and} \quad H_i = \left[\bigcup_{d\in D_i} (\Psi_{k^*} + d) \right] \setminus G_i. \tag{7}$$

2.4. Combinatorial lemma

Lemma 2.7. *Let p, q, and $\{l_i\}_{i\in\Omega}$ be positive integers with $\sum_{i\in\Omega} l_i = pq$. Suppose there exists an integer $r < p$ such that $l_i \leqslant r$ for all $i \in \Omega$ and $\#\{i\colon\ l_i = 1\} \geqslant rq$. Then there is a decomposition $\Omega = \bigcup_{s=1}^q \Omega_s$ such that for every $1 \leqslant s \leqslant q$, $\sum_{i\in\Omega_s} l_i = p$.*

Proof. It is trivial for $q = 1$. Suppose this inductive assumption on q is true for $q = 1, \ldots, (k-1)$.

Set $q = k$. Take $\Theta \subset \{i\colon\ l_i = 1\}$ with $\#(\Theta) = rk$ and select a maximal subset Δ_1 of $\Omega \backslash \Theta$ such that $\sum_{i\in\Delta_1} l_i < p$, we conclude that $\sum_{i\in\Delta_1} l_i \geqslant p - r$. Otherwise, $\sum_{i\in\Delta_1} l_i < p - r$, take $i_0 \in (\Omega\backslash\Delta_1)\backslash\Theta$, then $\sum_{i\in\Delta_1\cup\{i_0\}} l_i < (p-r) + r \leqslant p$, which contradicts the maximality of Δ_1. Now, $p - r \leqslant \sum_{i\in\Delta_1} l_i < p$. Choose subset Θ_1 of Θ such that $\#(\Theta_1) = p - \sum_{i\in\Delta_1} l_i \leqslant r$. Set $\Omega_1 = \Delta_1 \cup \Theta_1$, then $\sum_{i\in\Omega_1} l_i = p$. Applying inductive assumption on $(k-1)$ to $\Omega' = \Omega\backslash\Omega_1$ and $\#\{i \in \Omega'\colon\ l_i = 1\} \geqslant r(k-1)$, we get a decomposition $\Omega' = \bigcup_{s=2}^q \Omega_s$ with $\sum_{i\in\Omega_s} l_i = p$ $(s \geqslant 2)$. Therefore, the assumption for $q = k$ is true. □

3. Proof of Theorem 1.1

In fact, Theorem 1.1 is a consequence to the following theorem:

Theorem 3.1. *Suppose $A \subset \{0,1,\ldots,n-1\}^m$ with cardinality $\#A < n^m$. Let $E_A = \bigcup_{a\in A}(\frac{1}{n}E_A + \frac{a}{n})$ be the corresponding self-similar set in $\mathbb{R}^m$. Then the following statements are equivalent*:

(i) *E_A is totally disconnected*;
(ii) *E_A is of finite type*;
(iii) *E_A and $\Sigma_{n,\#A}$ are Lipschitz equivalent.*

Proof. $(\mathbf{1}) \Rightarrow (\mathbf{2})$: By Lemma 2.4, there exists an integer k_0 such that for any connected component χ of Ξ_{k_0} with $\chi \cap [0,1]^m \neq \emptyset$, χ is contained in $(-1,2)^m$. This means that, for all integer $k \geqslant 1$, any connected component of $\Psi_{(k+1)k_0}$ contains at most $(3n^{k_0})^m$ cubes of side $n^{-(k+1)k_0}$. By Remark 1, E_A is of finite type.

$(\mathbf{2}) \Rightarrow (\mathbf{3})$: Let k^* be the integer defined in Section 2.3. Since $\Sigma_{n,\#A} \simeq \Sigma_{n^{k^*},(\#A)^{k^*}}$, it suffices to show that $E_A \simeq \Sigma_{n^{k^*},(\#A)^{k^*}}$.

Suppose there are finitely many types $\{D_0, \ldots, D_{N(A)}\}$. For any connected component χ of Ψ_{k^*}, there is a set $D(\chi) \subset \mathbb{Z}^m$ such that $\chi = \frac{1}{n^{k^*}} \bigcup_{d\in D(\chi)}([0,1]^m + d)$. By replacing $[0,1]^m$ by E_A in (7), we define $\bar{G}_i$ in analogy to G_i:

$$\bar{G}_i = \bigcup_{z\in\{0,1\}^m} \left[\frac{1}{n^{k^*}} \bigcup_{\chi\in\bar{\Delta}_{z,k}} \bigcup_{d\in D(\chi)} (E_A + d) + d_{z,i} \right]. \tag{8}$$

We also set $\bar{H}_i = [\bigcup_{d\in D_i}(E_A + d)] \setminus \bar{G}_i$. Then for every type D_i, we get a compact set

$$F_i = \bigcup_{d \in D_i} (E_A + d) = \bar{G}_i \cup \bar{H}_i. \tag{9}$$

Set $\lambda(\bar{G}_i) = 1$ and $\lambda(\bar{H}_i) = \#D_i - 1$ for each i. Then $\lambda(\bar{G}_i), \lambda(\bar{H}_i) \leqslant M$.

Set $\bar{K} \in \{\bar{G}_0, \ldots, \bar{G}_{N(A)}\} \cup \{\bar{H}_1, \ldots, \bar{H}_{N(A)}\}$. Now, for $\bar{K} = \bar{G}_i$ or $\bar{H}_i$, we set $K = G_i$ or H_i respectively. Then $\bar{K}$ consists of $\lambda(\bar{K})(\#A)^{k^*}$ small copies of E_A with ratio n^{-k^*}. Therefore, in the same way, K has at least $\lambda(\bar{K})(\#A)^{k^*}/M$ connected components, where each connected component contains at most M cubes of side n^{-k^*}. Every connected component can be written as $\frac{1}{n^{k^*}}[z + \bigcup_{d \in D_j}([0,1]^m + d)]$ with type D_j in $\{D_0, \ldots, D_{N(A)}\}$ and $z \in \mathbb{Z}^m$. It follows from (9) that

$$\frac{1}{n^{k^*}}\left[z + \bigcup_{d \in D_j} (E_A + d)\right] = \frac{1}{n^{k^*}}[z + \bar{G}_j] \cup \frac{1}{n^{k^*}}[z + \bar{H}_j]. \tag{10}$$

This means that $\bar{K}$ contains at least $\lambda(\bar{K})(\#A)^{k^*}/M$ pairwise disjoint parts of the form $\frac{1}{n^{k^*}}[z + \bar{G}_j]$ or $\frac{1}{n^{k^*}}[z + \bar{H}_j]$. Here by (5),

$$\lambda(\bar{K})(\#A)^{k^*}/M \geqslant \lambda(\bar{K})M. \tag{11}$$

By setting $q = \lambda(\bar{K})$, $p = (\#A)^{k^*}$ and $r = M < p$ and applying (11) to Lemma 2.7, we get the decomposition

$$\bar{K} = \bar{K}_1 \cup \bar{K}_2 \cup \cdots \cup \bar{K}_{\lambda(\bar{K})}, \tag{12}$$

where

$$\bar{K}_i = \frac{1}{n^{k^*}} \bigcup_{s \in \Omega(K,i)} [z_s + L_s] \tag{13}$$

with $z_s \in \mathbb{Z}^m$ and $L_s \in \{\bar{G}_0, \ldots, \bar{G}_{N(A)}\} \cup \{\bar{H}_1, \ldots, \bar{H}_{N(A)}\}$ satisfying

$$\sum_{s \in \Omega(K,i)} \lambda(L_s) = (\#A)^{k^*}. \tag{14}$$

Therefore, $\{\bar{G}_0, \ldots, \bar{G}_{N(A)}\} \cup \{\bar{H}_1, \ldots, \bar{H}_{N(A)}\}$ are dust-like graph-directed sets satisfying (12)–(14).

For integers α, β with $1 \leqslant \alpha \leqslant \beta \leqslant (\#A)^{k^*}$, a subset Σ_α^β of $\Sigma_{n^{k^*},(\#A)^{k^*}}$ is defined by $\Sigma_\alpha^\beta = \{x_1 x_2 \cdots\colon\ x_1 \in \mathbb{N} \cap [\alpha, \beta]\}$. Take $\gamma * \Sigma_\alpha^\beta = \{x_1 x_2 x_3 \cdots\colon\ x_1 = \gamma$ and $x_2 x_3 \cdots \in \Sigma_\alpha^\beta\}$. Then there is a natural similitude from Σ_α^β to $\gamma * \Sigma_\alpha^\beta$ with ratio n^{-k^*}. Note that

$$\Sigma_1^{\beta-\alpha+1},\ \Sigma_\alpha^\beta \text{ are isometric},\ \ \Sigma_1^\alpha = \Sigma_1^1 \cup \Sigma_2^2 \cup \cdots \cup \Sigma_\alpha^\alpha, \tag{15}$$

and for a sequence $\{\lambda_1, \ldots, \lambda_t\}$ with $\lambda_1 + \cdots + \lambda_t = (\#A)^{k^*}$,

$$\Sigma_1^1 = \bigcup_{j=1}^{t} \left(1 * \Sigma_{\lambda_1 + \cdots + \lambda_{j-1} + 1}^{\lambda_1 + \cdots + \lambda_j}\right) \tag{16}$$

where there is a natural similitude from $\Sigma_1^{\lambda_j}$ to $1 * \Sigma_{\lambda_1 + \cdots + \lambda_{j-1} + 1}^{\lambda_1 + \cdots + \lambda_j}$ with ratio n^{-k^*}.

By (15)–(16), we obtain that

$$\{\overbrace{\Sigma_1^1, \ldots, \Sigma_1^1}^{N(A)+1}\} \cup \{\Sigma_1^{\lambda(\bar{H}_1)}, \Sigma_1^{\lambda(\bar{H}_2)}, \ldots, \Sigma_1^{\lambda(\bar{H}_{N(A)})}\}$$

are dust-like graph-directed sets on the graph determined by (12)–(14) and each similitude has ratio n^{-k^*}. Here $\Sigma_1^{\lambda(\bar{G}_i)} = \Sigma_1^1$ for all $0 \leqslant i \leqslant N(A)$.

By Lemma 2.2, $\bar{G}_0 (= E_A)$ and Σ_1^1 are Lipschitz equivalent. Thus E_A and $\Sigma_{n^{k^*},(\#A)^{k^*}}$ are Lipschitz equivalent.

(**3**) $\Rightarrow$ (**1**): This is obvious. □

Proof of Theorem 1.2. This theorem follows from Lemma 2.1 and (iii) of Theorem 3.1. □

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References

[1] D. Cooper, T. Pignataro, On the shape of Cantor sets, J. Differential Geom. 28 (1988) 203–221.
[2] G. David, S. Semmes, Fractured Fractals and Broken Dreams: Self-Similar Geometry through Metric and Measure, Oxford Univ. Press, 1997.
[3] K.J. Falconer, D.T. Marsh, Classification of quasi-circles by Hausdorff dimension, Nonlinearity 2 (1989) 489–493.
[4] K.J. Falconer, D.T. Marsh, On the Lipschitz equivalence of Cantor sets, Mathematika 39 (1992) 223–233.
[5] R.D. Mauldin, S.C. Williams, Hausdorff dimension in graph directed constructions, Trans. Amer. Math. Soc. 309 (1988) 811–829.
[6] H. Rao, H.-J. Ruan, L.-F. Xi, Lipschitz equivalence of self-similar sets, C. R. Acad. Sci. Paris, Ser. I 342 (2006) 191–196.
[7] Z.-Y. Wen, L.-F. Xi, Relations among Whitney sets, self-similar arcs and quasi-arcs, Israel J. Math. 136 (2003) 251–267.
[8] L.-F. Xi, Lipschitz equivalence of self-conformal sets, J. London Math. Soc. (2) 70 (2004) 369–382.
[9] L.-F. Xi, Quasi-Lipschitz equivalence of fractals, Israel J. Math. 160 (2007) 1–21.