C. R. Acad. Sci. Paris, Ser. I 348 (2010) 145–148

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Complex Analysis

Oka maps

Les applications d'Oka

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A R T I C L E   I N F O

Article history:
Received 11 November 2009
Accepted 7 December 2009
Available online 30 December 2009

Presented by Mikhaël Gromov

A B S T R A C T

We prove that for a holomorphic submersion of reduced complex spaces, the basic Oka property implies the parametric Oka property. It follows that a stratified subelliptic submersion, or a stratified fiber bundle whose fibers are Oka manifolds, enjoys the parametric Oka property.

R É S U M É

Nous prouvons que, pour une submersion holomorphe des espaces complexes réduits, la propriété d'Oka simple implique la propriété d'Oka paramétrique. En particulier, toute submersion sous-elliptique stratifié possède la propriété d'Oka paramétrique.

1. Oka properties of holomorphic maps

Let E and B be reduced complex spaces. A holomorphic map $\pi : E \to B$ is said to enjoy the *Basic Oka Property* (BOP) if, given a holomorphic map $f : X \to B$ from a reduced Stein space X and a continuous map $F_0 : X \to E$ satisfying $\pi \circ F_0 = f$ (a *lifting* of f) such that F_0 is holomorphic on a closed complex subvariety X' of X and in a neighborhood of a compact $\mathcal{O}(X)$-convex subset K of X, there is a homotopy of liftings $F_t : X \to E$ $(t \in [0,1])$ of f to a holomorphic lifting F_1 such that for every $t \in [0,1]$, F_t is holomorphic in a neighborhood of K (independent of t), $\sup_{x \in K} \operatorname{dist}(F_t(x), F_0(x)) < \epsilon$, and $F_t|_{X'} = F_0|_{X'}$ (the homotopy is fixed on X').

By definition, a complex manifold Y enjoys BOP if and only if the trivial map $Y \to$ *point* does. This is equivalent to several other properties, from the simplest *Convex Approximation Property* (CAP) to the *Parametric Oka Property* (POP) concerning compact families of maps from reduced Stein spaces to Y [2]. A complex manifold enjoying these equivalent properties is called an *Oka manifold* [2,11]; these are precisely the *fibrant complex manifolds* in Lárusson's model category [9]. Here we prove that BOP $\Rightarrow$ POP also holds for holomorphic submersions. (The submersion condition corresponds to requiring smoothness as part of the definition of a variety being Oka. The singular case is rather problematic.)

Theorem 1.1. *For every holomorphic submersion $\pi : E \to B$ of reduced complex spaces, the basic Oka property implies the parametric Oka property.*

Recall [9] that a holomorphic map $\pi : E \to B$ enjoys the *Parametric Oka Property* (POP) if for any triple (X, X', K) as above and for any pair $P_0 \subset P$ of compact subsets in an Euclidean space $\mathbb{R}^m$ the following holds. Given a continuous

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1631-073X/$ – see front matter
doi:10.1016/j.crma.2009.12.004

map $f : P \times X \to B$ that is *X-holomorphic* (that is, $f(p,\cdot) : X \to B$ is holomorphic for every $p \in P$) and a continuous map $F_0 : P \times X \to E$ such that (a) $\pi \circ F_0 = f$, (b) $F_0(p,\cdot)$ is holomorphic on X for all $p \in P_0$ and is holomorphic on $K \cup X'$ for all $p \in P$, there exists for every $\epsilon > 0$ a homotopy of continuous liftings $F_t : P \times X \to E$ of f to an X-holomorphic lifting F_1 such that the following hold for all $t \in [0,1]$:

(i) $F_t = F_0$ on $(P_0 \times X) \cup (P \times X')$, and
(ii) F_t is X-holomorphic on K and $\sup_{p\in P,\, x\in K} \mathrm{dist}(F_t(p,x), F_0(p,x)) < \epsilon$.

A stratified subelliptic holomorphic submersion, or a stratified fiber bundle with Oka fibers, enjoys BOP [3,4]. Hence Theorem 1.1 implies:

Corollary 1.2.

(i) *Every stratified subelliptic submersion enjoys POP.*
(ii) *Every stratified holomorphic fiber bundle with Oka fibers enjoys POP.*

If $\pi : E \to B$ enjoys the Oka property then by considering liftings of constant maps $X \to b \in B$ we see that every fiber $E_b = \pi^{-1}(b)$ is an Oka manifold. For stratified fiber bundles the converse holds by Corollary 1.2.

Question: *Does every holomorphic submersion with Oka fibers enjoys the Oka property?*

A holomorphic map is said to be an *Oka map* if it is a topological (Serre) fibration and it enjoys POP. Such maps are *intermediate fibrations* in Lárusson's model category [9,10]. Corollary 1.2 implies:

Corollary 1.3.

(i) *Every holomorphic fiber bundle projection with Oka fiber is an Oka map.*
(ii) *A stratified subelliptic submersion, or a stratified holomorphic fiber bundle with Oka fibers, is an Oka map if and only if it is a Serre fibration.*

Corollary 1.2(i) and the proof by Ivarsson and Kutzschebauch [8] give the following solution of the parametric Gromov–Vaserstein problem [7,12].

Theorem 1.4. *Assume that X is a finite-dimensional reduced Stein space, P is a compact subset of $\mathbb{R}^m$, and $f : P \times X \to \mathrm{SL}_n(\mathbb{C})$ is a null-homotopic X-holomorphic mapping. Then there exist a natural number N and X-holomorphic mappings $G_1, \ldots, G_N : P \times X \to \mathbb{C}^{n(n-1)/2}$ such that*

$$f(p,x) = \begin{pmatrix} 1 & 0 \\ G_1(p,x) & 1 \end{pmatrix} \begin{pmatrix} 1 & G_2(p,x) \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & G_N(p,x) \\ 0 & 1 \end{pmatrix}$$

is a product of upper and lower diagonal unipotent matrices.

2. Reduction of Theorem 1.1 to an approximation property

Assume that $\pi : E \to B$ enjoys BOP and that $(X, X', K, P, P_0, f, F_0)$ are as in the definition of POP, with $P_0 \subset P \subset \mathbb{R}^m \subset \mathbb{C}^m$. Set

$$Z = \mathbb{C}^m \times X \times E, \qquad Z_0 = \mathbb{C}^m \times X \times B, \qquad \psi = (\mathrm{id}_{\mathbb{C}^m \times X}) \times \pi : Z \to Z_0. \tag{1}$$

Observe that ψ enjoys BOP (resp. POP) if and only if π does. To the map $f : P \times X \to B$ we associate the X-holomorphic section

$$g : P \times X \to Z_0, \qquad g(p,x) = \big(p, x, f(p,x)\big) \quad (p \in P,\ x \in X), \tag{2}$$

and to the π-lifting $F_0 : P \times X \to E$ of f we associate the section

$$G_0 : P \times X \to Z, \qquad G_0(p,x) = \big(p, x, F_0(p,x)\big) \quad (p \in P,\ x \in X). \tag{3}$$

Then $\psi \circ G_0 = g$, G_0 is X-holomorphic over $K \cup X'$, and $G_0|_{P_0 \times X}$ is X-holomorphic. We must find a homotopy $G_t : P \times X \to Z$ ($t \in [0,1]$) such that $\psi \circ G_t = g$ for all $t \in [0,1]$, G_1 is X-holomorphic, and for all $t \in [0,1]$ the map G_t has the same properties as G_0, G_t is uniformly close to G_0 on $K \times P$, and $G_t = G_0$ on $(P_0 \times X) \cup (P \times X')$. Set

$$Q = [0,1] \times P, \qquad Q_0 = \big(\{0\} \times P\big) \cup \big([0,1] \times P_0\big).$$

The following result is the key to the proof of Theorem 1.1.

Proposition 2.1. *If the submersion $\psi : Z \to Z_0$ (1) enjoys the basic Oka property, then it also enjoys the following Parametric Homotopy Approximation Property (PHAP): Let $K \subset L$ be compact $\mathcal{O}(X)$-convex subsets and let $U \supset K$, $V \supset L$ be open neighborhoods in X. Assume that $g : P \times V \to Z_0$ is an X-holomorphic section of the form (2) and $G_t : P \times V \to Z$ $(t \in [0,1])$ is a homotopy of sections (3) satisfying*

(a) $\psi \circ G_t = g$ *for* $t \in [0,1]$,
(b) $G_t(p,\cdot)$ *is holomorphic on* U *for* $(t,p) \in Q$, *and*
(c) $G_t(p,\cdot) = G_0(p,\cdot)$ *for* $(t,p) \in Q_0$, *and these are holomorphic on* V.

Let $\epsilon > 0$. After shrinking the neighborhoods $U \supset K$ and $V \supset L$, there exists a homotopy $\widetilde{G}_t : P \times V \to Z$ $(t \in [0,1])$ of the form (3) such that

(i) $\psi \circ \widetilde{G}_t = g$ *for all* $t \in [0,1]$,
(ii) *for each* $(t,p) \in Q$ *the map* $\widetilde{G}_t(p,\cdot) : V \to Z$ *is holomorphic and it satisfies* $\sup_{x\in K} \mathrm{dist}(\widetilde{G}_t(p,x), G_t(p,x)) < \epsilon$, *and*
(iii) $\widetilde{G}_t(p,\cdot) = G_t(p,\cdot)$ *for each* $(t,p) \in Q_0$.

Furthermore, there is a homotopy from $\{G_t\}$ to $\{\widetilde{G}_t\}$ consisting of homotopies with the same properties as $\{G_t\}$.

For families of sections of a holomorphic submersion $\pi : Z \to X$ over a Stein space X, PHAP holds if $Z \to X$ admits a fiber-dominating spray over a neighborhood of L [7,5], or a finite fiber-dominating family of sprays [1]. Submersions admitting such sprays over small open subsets of X are called *elliptic*, resp. *subelliptic*. If PHAP holds over small open subsets of X then sections $X \to Z$ satisfy the parametric Oka property (Gromov [7, Theorem 4.5]; the details can be found in [3,5]). The same proof applies in our situation (see [4, Theorem 4.2]) and shows that PHAP implies Theorem 1.1.

3. Proof of Proposition 2.1

Let $h : \mathcal{E} \to Z$ denote the holomorphic vector bundle whose fiber over a point $z \in Z$ equals the tangent space at z to the (smooth) fiber of ψ. The restriction $\mathcal{E}|_\Omega$ to any open Stein domain $\Omega \subset Z$ is a reduced Stein space. By standard techniques we obtain for every such Ω an open Stein neighborhood $W \subset \mathcal{E}|_\Omega$ of the zero section $\Omega \subset \mathcal{E}|_\Omega$, with W Runge in $\mathcal{E}|_\Omega$, and a continuous map $s : \mathcal{E}|_\Omega \to Z$ satisfying $\psi \circ s = \psi \circ h$, such that s is the identity on the zero section, it is holomorphic on W, and it maps the fiber $W_z = \mathcal{E}_z \cap W$ over a point $z \in Z$ biholomorphically onto a neighborhood of the point $z = s(0_z)$ in the fiber $Z_{\psi(z)} = \psi^{-1}(\psi(x))$. Such s is a fiber-dominating spray in the sense of [7], except that it is not globally holomorphic.

By [6, Corollary 2.2] each of the sets

$$S_0 = G_0(P \times L) \subset Z, \qquad \Sigma_t = G_t(P \times K) \subset Z \quad \big(t \in [0,1]\big)$$

is a Stein compactum in Z. By compactness of $\bigcup_{t\in[0,1]} \Sigma_t$ there exist numbers $0 = t_0 < t_1 < \cdots < t_N = 1$, Stein domains $\Omega_0, \ldots, \Omega_{N-1} \subset Z$ satisfying

$$\bigcup_{t\in[t_j,t_{j+1}]} \Sigma_t \subset \Omega_j \quad (j = 0,1,\ldots,N-1), \tag{4}$$

and for every $j = 0,1,\ldots,N-1$ there exist an open Stein neighborhood $W_j \subset \mathcal{E}|_{\Omega_j}$ of the zero section Ω_j of $\mathcal{E}|_{\Omega_j}$ such that W_j is Runge in $\mathcal{E}|_{\Omega_j}$ and has convex fibers, a fiber-spray $s_j : \mathcal{E}|_{\Omega_j} \to Z$ as above that is holomorphic on W_j, and a homotopy ξ_t $(t \in [t_j, t_{j+1}])$ of X-holomorphic sections of the restricted bundle $\mathcal{E}|_{G_{t_j}(P\times U)}$, with the range contained in W_j, such that

(i) ξ_{t_j} is the zero section of $\mathcal{E}|_{G_{t_j}(P\times U)}$,
(ii) $\xi_t(p,\cdot)$ is the zero section when $p \in P_0$ and $t \in [t_j, t_{j+1}]$, and
(iii) $s_j \circ \xi_t \circ G_{t_j} = G_t$ on $P \times U$ for all $t \in [t_j, t_{j+1}]$.

(See Fig. 1.) For a given collection (Ω_j, W_j, s_j) the existence of homotopies ξ_t is stable under sufficiently small perturbations of the homotopy G_t.

Consider the homotopy of sections $\{\xi_t\}_{t\in[0,t_1]}$ of $\mathcal{E}|_{G_0(P\times U)}$. By the parametric version of the Oka–Weil theorem we can approximate ξ_t uniformly on $P \times K$ by X-holomorphic sections $\widetilde{\xi}_t$ of $\mathcal{E}|_{G_0(P\times V')}$ for an open set $V' \subset X$ with $L \subset V' \subset V$. Further, we may choose $\widetilde{\xi}_t = \xi_t$ for $t = 0$ and on $P_0 \times V'$. In the sequel the set V' may shrink around L.

By [6, Corollary 2.2] there is an open Stein neighborhood Ω of S_0 in Z such that S_0 is $\mathcal{O}(\Omega)$-convex. Hence $\Sigma_0 = G_0(P \times K) \subset S_0$ is also $\mathcal{O}(\Omega)$-convex, and it follows that $W_0 \cap \mathcal{E}|_{\Sigma_0}$ is exhausted by $\mathcal{O}(\mathcal{E}|_\Omega)$-convex compact sets. Since $\mathcal{E}|_\Omega$ is a reduced Stein space and s_0 extends continuously to $\mathcal{E}|_\Omega$ preserving the property $\psi \circ s_0 = \psi \circ h$, the assumed BOP of ψ implies that s_0 can be approximated on the range of the homotopy $\{\xi_t\colon t \in [0,t_1]\}$ (which is contained in $W_0 \cap \mathcal{E}|_{\Sigma_0}$) by a holomorphic map $\widetilde{s}_0 : \mathcal{E}|_\Omega \to Z$ which equals the identity on the zero section and satisfies $\psi \circ \widetilde{s}_0 = \psi \circ h$. The homotopy

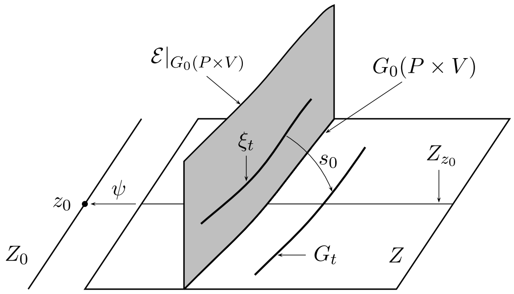

Fig. 1. Lifting sections G_t to the spray bundle $\mathcal{E}|_{G_0(P\times V)}$.

$$\widetilde{G}_t = \widetilde{s}_0 \circ \widetilde{\xi}_t \circ G_0 : P \times V' \to Z \quad \big(t \in [0, t_1]\big)$$

is fixed over P_0, X-holomorphic on V', $\widetilde{G}_0 = G_0$, and $\widetilde{G}_t$ approximates G_t uniformly on $P \times K$ (also uniformly with respect to $t \in [0, t_1]$). If the approximation is sufficiently close, we obtain a new homotopy $\{G_t\}_{t\in[0,1]}$ that agrees with $\widetilde{G}_t$ for $t \in [0, t_1]$ (hence is X-holomorphic on L), and that agrees with the initial homotopy for $t \in [t_1', 1]$ for some $t_1' > t_1$ close to t_1.

We now repeat the same argument with the parameter interval $[t_1, t_2]$ using G_{t_1} as the initial reference map. This produces a new homotopy that is X-holomorphic on L for all values $t \in [0, t_2]$. After N steps of this kind we obtain a homotopy satisfying the conclusion of Proposition 2.1.

Acknowledgements

I wish to thank Finnur Lárusson for his helpful remarks on a preliminary version of the paper. Research was supported in part by grants P1-0291 and J1-2043-0101 from ARRS, Republic of Slovenia.

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