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Probability Theory

BSDEs with random default time and related zero-sum stochastic differential games

EDSR avec le temps aléatoire de défaut et les jeux associés différentiels stochastiques à somme nulle

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ABSTRACT

In this Note we are concerned with backward stochastic differential equations with random default time. The equations are driven by Brownian motion as well as a mutually independent martingale appearing in a defaultable setting. We show that these equations have unique solutions and a comparison theorem for their solutions. As an application, we get a saddle-point strategy for the related zero-sum stochastic differential game problem.

RÉSUMÉ

Dans cette Note, nous considirons les équations différentielles stochastiques rétrogrades avec le temps aléatoire de défaut. Les équations sont dirigées par mouvement brownien ainsi qu'une martingale mutuellement indépendants apparaissant dans un cadre de defaut. Nous montrons que ces équations ont des solutions uniques et un théorème de comparaison pour leurs solutions. Comme une application, nous obtenons une stratégie de point-selle pour le jeu associé différentiel stochastique à somme nulle.

Version française abrégée

Dans cette Note, nous modelons un risque de défaut utilisant l'EDSR avec le temps aléatoire de défaut, voir (1) pour la forme générale d'EDSR avec le temps aléatoire de défaut. Nous montrons les résultats suivants.

Théorème 0.1. *Supposons que g satisfait* (a) *et* (b)*, alors pour toute condition terminale* $\xi(H_T) \in L^2(\mathcal{G}_T;\mathbb{R}^m)$*, EDSR* (1) *a une solution unique, c.à-d., il existe un triple unique de processus* $\mathcal{G}_t$*-adaptés*

$$(Y_t, Z_t, \zeta_t) \in S^2_{\mathcal{G}}\big(0,T;\mathbb{R}^m\big) \times L^2_{\mathcal{G}}\big(0,T;\mathbb{R}^{m\times d}\big) \times L^{2,\tau}_{\mathcal{G}}\big(0,T;\mathbb{R}^{m\times k}\big)$$

satisfaisant (1)*.*

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Théorème 0.2. *Supposons que ξ, $\bar{\xi}$ satisfont les même conditions que dans Théorème* 0.1, *g satisfait* (a), $\bar{g}_s \in L^2_{\mathcal{G}}(0,T;\mathbb{R})$. *Soit* (Y,Z,ζ), $(\bar{Y},\bar{Z},\bar{\zeta})$ *les solutions uniques de* (2), (3) *respectivement. Si*

$$\xi \geqslant \bar{\xi}, \quad g(t,\bar{Y}_t,\bar{Z}_t,\bar{\zeta}_t) \geqslant \bar{g}_t, \quad \text{a.e., a.s.},$$

alors

$$Y_t \geqslant \bar{Y}_t, \quad \text{a.e., a.s.}$$

De plus, le texte suivant est vrai (*le théorème de comparaison stricte*) :

$$Y_0 = \bar{Y}_0 \quad \Leftrightarrow \quad \xi = \bar{\xi}, \quad g(t,\bar{Y}_t,\bar{Z}_t,\bar{\zeta}_t) \equiv \bar{g}_t.$$

Comme une application des résultats ci-dessus, nous obtenons une stratégie de point-selle pour le jeu associé différentiel stochastique à somme nulle dans un cadre de defaut.

1. Introduction

Credit risk is a kind of the most dangerous financial risk. Particularly in recent years it has been greatly concerned once more. The most extensively studied form of credit risk is the default risk, that is, the risk that a counterpart in a financial contract will not fulfill a contractual commitment to meet her/his obligations stated in the contract. Many people, such as Bielecki, Jarrow, Jeanblanc, Kusuoka and so on, have worked on this subject (see e.g. [2,3,6,8,9]).

It is well known that, in the framework of Brownian filtration, the general form of BSDE was firstly studied by Pardoux and Peng [11]. Since then, the theory of BSDEs has been studied with great interest. One of the achievements of this theory is the comparison theorem. It is due to Peng [13] and then generalized by Pardoux and Peng [12], El Karoui et al. [5]. It allows to compare the solutions of two BSDEs whenever we can compare the terminal conditions and the generators. With these results, BSDEs can be applied to solve stochastic differential game problems, see e.g. [4,7].

In this Note, we present a new approach (i.e., BSDE with random default time) to model default risk. In fact, this type of BSDEs appears very naturally when dealing with the hedging/pricing problem (see [14]) and the utility maximization problem (see Bielecki et al. [2]) in a defaultable setting. For these equations, we prove an existence and uniqueness result as well as a comparison theorem. It should be noted here that, the comparison theorem needs one more condition for the generator than the existence and unique theorem, which is different from the classical case.

The Note is organized as follows: in Section 2, we list some notations and assumptions. In Section 3, we mainly study BSDEs with random default time. As an application, in Section 4 we solve a zero-sum stochastic differential game problem in a defaultable setting.

2. Notations and assumptions

Let $\{B_t;\ t \geqslant 0\}$ be a d-dimensional standard Brownian motion on a probability space $(\Omega,\mathcal{F},P)$ and $(\mathcal{F}_t)_{t\geqslant 0}$ be its natural filtration. Denote by $|\cdot|$ the norm in $\mathbb{R}^m$.

Let $\{\tau_i;\ i=1,2,\ldots,k\}$ be k nonnegative random variables satisfying

$$P(\tau_i > 0) = 1, \qquad P(\tau_i > t) > 0, \quad \forall t \in \mathbb{R}_+, \qquad P(\tau_i = \tau_j) = 0 \quad (i \neq j).$$

For each i, we introduce a right-continuous process $\{H^i_t;\ t \geqslant 0\}$ by setting $H^i_t := 1_{\{\tau_i \leqslant t\}}$ and denote by $\mathbb{H}^i = (\mathcal{H}^i_t)_{t\geqslant 0}$ the associated filtration $\mathcal{H}^i_t = \sigma(H^i_s\colon\ 0 \leqslant s \leqslant t)$.

Just as in the general reduced-form approach, for fixed $T>0$, there are two kinds of information: one from the assets prices, denoted by $\mathbb{F} = (\mathcal{F}_t)_{0\leqslant t\leqslant T}$, and the other from the default times $\{\tau_i;\ i=1,2,\ldots,k\}$, denoted by $\{\mathbb{H}^i;\ i=1,2,\ldots,k\}$ from the above. The enlarged filtration considered is denoted by $\mathbb{G} = (\mathcal{G}_t)_{0\leqslant t\leqslant T}$ where $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}^1_t \vee \mathcal{H}^2_t \vee \cdots \vee \mathcal{H}^k_t$, which indicates that each τ_i is a $\mathbb{G}$-stopping time but not necessarily an $\mathbb{F}$-stopping time in the general case.

Now we assume the following (see [9]):

(**A**) There exist $\mathbb{F}$-adapted processes $\gamma^i \geqslant 0$ $(i=1,2,\ldots,k)$ such that $M^i_t := H^i_t - \int_0^t 1_{\{\tau_i > s\}} \gamma^i_s \,\mathrm{d}s$ $(i=1,2,\ldots,k)$ are $\mathbb{G}$-martingales under P.

(**H**) Every $\mathbb{F}$-local martingale is a $\mathbb{G}$-local martingale.

It should be mentioned that (**H**) is a very general and essential hypothesis in the area of enlarged filtration (see [10]).

The following are just for the sake of simplicity:

(i) notations of vectors:
$H_t := (H^1_t, H^2_t, \ldots, H^k_t)'$, $M_t := (M^1_t, M^2_t, \ldots, M^k_t)'$, $1_{\{\tau>t\}}\gamma_t := (1_{\{\tau_1>t\}}\gamma^1_t, 1_{\{\tau_2>t\}}\gamma^2_t, \ldots, 1_{\{\tau_k>t\}}\gamma^k_t)'$, where $(\cdot)'$ is the transpose;

(ii) notations of sets: $L^2(\mathcal{G}_T;\mathbb{R}^m) := \{\xi \in \mathbb{R}^m \mid \xi$ is a $\mathcal{G}_T$-measurable random variable such that $E|\xi|^2 < +\infty\}$,
$L^2_{\mathcal{G}}(0,T;\mathbb{R}^m) := \{\varphi : \Omega \times [0,T] \to \mathbb{R}^m \mid \varphi$ is $\mathcal{G}_t$-progressively measurable and $E\int_0^T |\varphi_t|^2 \,\mathrm{d}t < +\infty\}$,
$S^2_{\mathcal{G}}(0,T;\mathbb{R}^m) := \{\varphi : \Omega \times [0,T] \to \mathbb{R}^m \mid \varphi$ is $\mathcal{G}_t$-progressively measurable and $E[\sup_{0\leqslant t\leqslant T} |\varphi_t|^2] < +\infty\}$,

$L^{2,\tau}_{\mathcal{G}}(0,T;\mathbb{R}^{m\times k}) := \{\varphi : \Omega \times [0,T] \to \mathbb{R}^{m\times k} \mid \varphi$ is progressively measurable and $E\int_0^T |\varphi_t|^2 1_{\{\tau>t\}}\gamma_t\,\mathrm{d}t := E\int_0^T \sum_{j=1}^m \times \sum_{i=1}^k |\varphi_{ji,t}|^2 1_{\{\tau_i>t\}}\gamma_t^i\,dt < +\infty\}$.

3. BSDE with random default time

This section discusses BSDEs with random default time of the following general form:

$$Y_t = \xi(H_T) + \int_t^T g(s, Y_s, Z_s, \zeta_s)\,\mathrm{d}s - \int_t^T Z_s\,\mathrm{d}B_s - \int_t^T \zeta_s\,\mathrm{d}M_s. \tag{1}$$

The function g is called the generator of (1). We assume that $g(\omega, t, y, z, \varsigma) : \Omega \times [0,T] \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times \mathbb{R}^{m\times k} \to \mathbb{R}^m$ satisfies the following conditions:

(a) $g(\cdot, 0, 0, 0) \in L^2_{\mathcal{G}}(0,T;\mathbb{R}^m)$;
(b) the Lipschitz condition: for each $(t, y, z, \varsigma), (t, \bar{y}, \bar{z}, \bar{\varsigma}) \in [0,T] \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times \mathbb{R}^{m\times k}$, there exists a constant $C \geqslant 0$ such that

$$\left|g(t,y,z,\varsigma) - g(t,\bar{y},\bar{z},\bar{\varsigma})\right| \leqslant C\left(|y-\bar{y}| + |z-\bar{z}| + |\varsigma - \bar{\varsigma}| 1_{\{\tau>t\}}\sqrt{\gamma_t}\right).$$

Our object is to find a triple $(Y_t, Z_t, \zeta_t) \in S^2_{\mathcal{G}}(0,T;\mathbb{R}^m) \times L^2_{\mathcal{G}}(0,T;\mathbb{R}^{m\times d}) \times L^{2,\tau}_{\mathcal{G}}(0,T;\mathbb{R}^{m\times k})$ satisfying (1). For this, we have the following existence and uniqueness theorem:

Theorem 3.1 *(Existence and Uniqueness Theorem). (See [14, Theorem 3.1].) Assume that g satisfies* (a) *and* (b)*, then for any fixed terminal condition $\xi(H_T) \in L^2(\mathcal{G}_T;\mathbb{R}^m)$, BSDE* (1) *has a unique solution, i.e., there exists a unique triple of $\mathcal{G}_t$-adapted processes*

$$(Y_t, Z_t, \zeta_t) \in S^2_{\mathcal{G}}\big(0,T;\mathbb{R}^m\big) \times L^2_{\mathcal{G}}\big(0,T;\mathbb{R}^{m\times d}\big) \times L^{2,\tau}_{\mathcal{G}}\big(0,T;\mathbb{R}^{m\times k}\big)$$

satisfying (1)*.*

Remark 3.2. The solution of (1) is unique, that is to say, if both (Y, Z, ζ) and $(\bar{Y}, \bar{Z}, \bar{\zeta}) \in S^2_{\mathcal{G}}(0,T;\mathbb{R}^m) \times L^2_{\mathcal{G}}(0,T;\mathbb{R}^{m\times d}) \times L^{2,\tau}_{\mathcal{G}}(0,T;\mathbb{R}^{m\times k})$ satisfy (1), then

$$E\int_0^T |Y_t - \bar{Y}_t|^2\,\mathrm{d}t = 0, \qquad E\int_0^T |Z_t - \bar{Z}_t|^2\,\mathrm{d}t = 0, \qquad E\int_0^T |\zeta_t - \bar{\zeta}_t|^2 1_{\{\tau>t\}}\gamma_t\,\mathrm{d}t = 0.$$

Consider the following two 1-dimensional BSDEs with random default time:

$$Y_t = \xi(H_T) + \int_t^T g(s, Y_s, Z_s, \zeta_s)\,\mathrm{d}s - \int_t^T Z_s\,\mathrm{d}B_s - \int_t^T \zeta_s\,\mathrm{d}M_s, \tag{2}$$

$$\bar{Y}_t = \bar{\xi}(H_T) + \int_t^T \bar{g}_s\,\mathrm{d}s - \int_t^T \bar{Z}_s\,\mathrm{d}B_s - \int_t^T \bar{\zeta}_s\,\mathrm{d}M_s. \tag{3}$$

For the generator function g, we introduce one more assumption:

(c) for each $(t, y, z) \in [0,T] \times \mathbb{R} \times \mathbb{R}^d$, $(\varsigma, \bar{\varsigma}) \in \mathbb{R}^k \times \mathbb{R}^k$, $(\varsigma^i - \bar{\varsigma}^i)1_{\{\tau_i>t\}}\gamma_t^i \neq 0$, the following holds:

$$\frac{g(t,y,z,\tilde{\varsigma}^{i-1}) - g(t,y,z,\tilde{\varsigma}^i)}{(\varsigma^i - \bar{\varsigma}^i)1_{\{\tau_i>t\}}\gamma_t^i} > -1,$$

where $\tilde{\varsigma}^i = (\bar{\varsigma}^1, \bar{\varsigma}^2, \ldots, \bar{\varsigma}^i, \varsigma^{i+1}, \varsigma^{i+2}, \ldots, \varsigma^k)$ and ς^i is the i-th component of ς.

Theorem 3.3 *(Comparison Theorem). (See [14, Theorem 3.2].) Suppose $\xi, \bar{\xi}$ satisfy the same assumptions as in Theorem* 3.1*, g satisfies* (a)–(c)*, $\bar{g}_s \in L^2_{\mathcal{G}}(0,T;\mathbb{R})$. Let (Y, Z, ζ), $(\bar{Y}, \bar{Z}, \bar{\zeta})$ be the unique solutions of* (2), (3) *respectively. If*

$$\xi \geqslant \bar{\xi}, \quad g(t, \bar{Y}_t, \bar{Z}_t, \bar{\zeta}_t) \geqslant \bar{g}_t, \quad \text{a.e., a.s.,}$$

then

$$Y_t \geqslant \bar{Y}_t, \quad \text{a.e., a.s.}$$

Besides, the following holds true (*the strict comparison theorem*):

$$Y_0 = \bar{Y}_0 \quad \Leftrightarrow \quad \xi = \bar{\xi}, \qquad g(t, \bar{Y}_t, \bar{Z}_t, \bar{\zeta}_t) \equiv \bar{g}_t.$$

Remark 3.4. Condition (c) for the generator g is significant for the comparison theorem. In [14], an example shows that the strict comparison theorem will not hold if g does not satisfy (c).

4. Application in zero-sum stochastic differential game problem

Assume here that $m = d = k$ and $\gamma_t \geqslant 0$ is bounded. Let $x_0 \in \mathbb{R}^m$ and let X_t be the unique solution of the following stochastic differential equation:

$$X_t = x_0 + \int_0^t \sigma(s, X_{s-})\,\mathrm{d}B_s + \int_0^t \kappa(s, X_{s-})\,\mathrm{d}M_s,$$

where the mapping $\sigma : [0,T] \times \mathbb{R}^m \to \mathbb{R}^{m\times m}$ and $\kappa : [0,T] \times \mathbb{R}^m \to \mathbb{R}^{m\times m}$ satisfy the following assumptions:

(i) for $1 \leqslant i, j \leqslant m$, σ_{ij} and κ_{ij} are progressively measurable;
(ii) for any $(t,x) \in [0,T] \times \mathbb{R}^m$, there exists a constant $C_1 \geqslant 0$ such that $|\sigma(t,x)| + |\kappa(t,x)| \leqslant C_1(1+|x|)$;
(iii) for any $(t,x), (t,y) \in [0,T] \times \mathbb{R}^m$, there exists a constant $C_2 \geqslant 0$ such that $|\sigma(t,x) - \sigma(t,y)| + |\kappa(t,x) - \kappa(t,y)| \leqslant C_2|x-y|$;
(iv) $\sigma(t,x)$, $\kappa(t,x)$ are invertible and $\sigma^{-1}(t,x)$, $\kappa^{-1}(t,x)$ are bounded.

Let U (resp. V) be a compact metric space and $\mathcal{U}$ (resp. $\mathcal{V}$) be the space of all progressively measurable processes $u = (u_t)_{t\in[0,T]}$ (resp. $v = (v_t)_{t\in[0,T]}$) with values in U (resp. V).

Let the drift function b map $[0,T] \times \mathbb{R}^m \times U \times V$ into $\mathbb{R}^m$. Furthermore, b is supposed to satisfy

(i) b is $\mathcal{B}([0,T] \times \mathbb{R}^m \times U \times V)$-measurable;
(ii) $b(t,x,u,v)$ is bounded for any (t,x,u,v);
(iii) for any $(t,x) \in [0,T] \times \mathbb{R}^m$, $b(t,x,\cdot,\cdot)$ is continuous on $U \times V$.

Now for each $u \in \mathcal{U}$, $v \in \mathcal{V}$, let $L^{u,v}$ be the positive local martingale solution of

$$L_t^{u,v} = 1 + \int_0^t L_{s-}^{u,v}\big(\sigma^{-1}(s, X_{s-})b(s, X_{s-}, u_s, v_s)\,\mathrm{d}B_s + \kappa^{-1}(s, X_{s-})c(s, X_{s-}, u_s, v_s)\,\mathrm{d}M_s\big),$$

where for any (t,x,u,v), $i = 1,2,\ldots,m$, the i-th component of $\kappa^{-1}(t,x)c(t,x,u,v)$ is larger than -1, i.e., $(\kappa^{-1}(t,x)c(t,x,u,v))^i > -1$.

According to the Girsanov Theorem (see [9, Proposition 3.1] or [3, Proposition 3.2.2]), $P^{u,v}$ defined by $\frac{\mathrm{d}P^{u,v}}{\mathrm{d}P}\big|\mathcal{G}_T = L_T^{u,v}$ is a probability measure equivalent to P. Moreover, under $P^{u,v}$, the process $B_t^{u,v} = B_t - \int_0^t \sigma^{-1}(s, X_{s-})b(s, X_{s-}, u_s, v_s)\,\mathrm{d}s$ is a Brownian motion, the processes $M_t^{i,u,v} = M_t^i - \int_0^t (\kappa^{-1}(s, X_{s-})c(s, X_{s-}, u_s, v_s))^i 1_{\{\tau_i > s\}}\gamma_s^i\,\mathrm{d}s$ $(i = 1,2,\ldots,m)$ are $\mathbb{G}$-martingales orthogonal to each other and orthogonal to $B_t^{u,v}$. Then $(X_t)_{0\leqslant t\leqslant T}$ satisfies

$$X_t = x_0 + \int_0^t \big(b(s, X_{s-}, u_s, v_s) + c(s, X_{s-}, u_s, v_s)1_{\{\tau > s\}}\gamma_s\big)\,\mathrm{d}s + \int_0^t \sigma(s, X_{s-})\,\mathrm{d}B_s^{u,v} + \int_0^t \kappa(s, X_{s-})\,\mathrm{d}M_s^{u,v}.$$

It means that $(X_t)_{0\leqslant t\leqslant T}$ is a weak solution for the above stochastic differential equation and it stands for an evolution of a controlled system.

It is well known that in zero-sum game problems, there are two players J_1 and J_2. We suppose that J_1 (resp. J_2) chooses a control $u(t,x) \in U$ (resp. $v(t,x) \in V$). Now we introduce two functions $f : [0,T] \times \mathbb{R}^m \times U \times V \to \mathbb{R}_+$, satisfying the same assumptions as b, and $h : \{0,1\} \times \mathbb{R}^m \to \mathbb{R}_+$ which is measurable, bounded. Let $E^{u,v}$ denote the expectation w.r.t. $P^{u,v}$. Then the cost functional, which is a cost (resp. reward) for J_1 (resp. J_2), corresponding to $u \in \mathcal{U}$ and $v \in \mathcal{V}$ is given by

$$J^{u,v} = E_{u,v}\left[\int_0^T f(s, X_s, u_s, v_s)\,\mathrm{d}s + h(H_T, X_T)\right].$$

The object of J_1 (resp. J_2) is to minimize (resp. maximize) the cost functional. In this zero-sum game problem, we aim at showing the existence of a saddle point, more precisely, a pair $(\tilde{u}^*, \tilde{v}^*)$ such that $J(\tilde{u}^*, v) \leqslant J(\tilde{u}^*, \tilde{v}^*) \leqslant J(u, \tilde{v}^*)$ for each $(u,v) \in \mathcal{U} \times \mathcal{V}$.

Thus let us define the Hamilton function associated with this game problem as following: $\forall(t,x,z,\varsigma,u,v)\in[0,T]\times\mathbb{R}^m\times\mathbb{R}^m\times\mathbb{R}^m\times U\times V$,

$$H(t,x,z,\varsigma,u,v):=z\sigma^{-1}(t,x)b(t,x,u,v)+\varsigma\kappa^{-1}(t,x)c(t,x,u,v)1_{\{\tau>t\}}\gamma_t+f(t,x,u,v).$$

Here we should pay special attention to the difference between the notations of the Hamilton function $H(t,\cdot,\cdot,\cdot,\cdot,\cdot)$ and the default process H_t.

Next assume that Isaacs' condition, which plays an important role in zero-sum stochastic differential game problems, is fulfilled, i.e., for any $(t,x,z,\varsigma)\in[0,T]\times\mathbb{R}^m\times\mathbb{R}^m\times\mathbb{R}^m$,

$$\inf_{u\in U}\sup_{v\in V}H(t,x,z,\varsigma,u,v)=\sup_{v\in V}\inf_{u\in U}H(t,x,z,\varsigma,u,v).$$

Under the above Isaacs' condition, through the assumptions above and Benes's selection theorem (see e.g. [1]), the following holds true (see e.g. [4]):

Proposition 4.1. *There exist two measurable functions $u^*(t,x,z,\varsigma)$, $v^*(t,x,z,\varsigma)$ mapping from $[0,T]\times\mathbb{R}^m\times\mathbb{R}^m\times\mathbb{R}^m$ into U, V respectively such that*

(i) *the pair $(u^*,v^*)(t,x,z,\varsigma)$ is a saddle point for the function H, i.e.,*

$$\begin{aligned}&H\big(t,x,z,\varsigma,u^*(t,x,z,\varsigma),v^*(t,x,z,\varsigma)\big)\leqslant H\big(t,x,z,\varsigma,u,v^*(t,x,z,\varsigma)\big),\quad\forall u\in U,\\&H\big(t,x,z,\varsigma,u^*(t,x,z,\varsigma),v^*(t,x,z,\varsigma)\big)\geqslant H\big(t,x,z,\varsigma,u^*(t,x,z,\varsigma),v\big),\quad\forall v\in V;\end{aligned}$$

(ii) *the function $(z,\varsigma)\to H(t,x,z,\varsigma,u^*(t,x,z,\varsigma),v^*(t,x,z,\varsigma))$ satisfies* (b) *and* (c)*, uniformly in (t,x).*

Now we introduce two notations just for simplicity:

$$H(t,z,\varsigma):=H(t,X_{t-},z,\varsigma,u_t,v_t);\qquad H^*(t,z,\varsigma):=H\big(t,X_{t-},z,\varsigma,u^*(t,X_{t-},z,\varsigma),v^*(t,X_{t-},z,\varsigma)\big).$$

Suppose that J_1 (resp. J_2) has chosen $u\in\mathcal{U}$ (resp. $v\in\mathcal{V}$). The conditional expected remaining cost from time $t\in[0,T]$ is

$$J_t^{u,v}=E_{u,v}^{\mathcal{G}_t}\left[\int_t^T f(s,X_s,u_s,v_s)\,\mathrm{d}s+h(H_T,X_T)\right].$$

It is obvious that $J_0^{u,v}=J^{u,v}$. The following theorem tells us that the conditional costs can be characterized as solutions of BSDEs with random default time:

Theorem 4.2. *(See [14, Theorem 4.1].) The BSDE with random default time*

$$Y_t=h(H_T,X_T)+\int_t^T H(s,Z_s,\zeta_s)\,\mathrm{d}s-\int_t^T Z_s\,\mathrm{d}B_s-\int_t^T \zeta_s\,\mathrm{d}M_s$$

has a unique solution $(Y,Z,\zeta)\in S^2_{\mathcal{G}}(0,T;\mathbb{R})\times L^2_{\mathcal{G}}(0,T;\mathbb{R}^m)\times L^{2,\tau}_{\mathcal{G}}(0,T;\mathbb{R}^m)$ satisfying $Y_t=J_t^{u,v}$.

Next is the main result of this part:

Theorem 4.3. *(See [14, Theorem 4.2].) The BSDE with random default time*

$$Y_t=h(H_T,X_T)+\int_t^T H^*(s,Z_s,\zeta_s)\,\mathrm{d}s-\int_t^T Z_s\,\mathrm{d}B_s-\int_t^T \zeta_s\,\mathrm{d}M_s$$

has a unique solution $(Y,Z,\zeta)\in S^2_{\mathcal{G}}(0,T;\mathbb{R})\times L^2_{\mathcal{G}}(0,T;\mathbb{R}^m)\times L^{2,\tau}_{\mathcal{G}}(0,T;\mathbb{R}^m)$ satisfying

$$Y_t=J_t^{\tilde{u}^*,\tilde{v}^*},$$

where $\tilde{u}^*(t,X_{t-})=u^*(t,X_{t-},Z_t,\zeta_t)$, $\tilde{v}^*(t,X_{t-})=v^*(t,X_{t-},Z_t,\zeta_t)$. Moreover, the pair $(\tilde{u}^*,\tilde{v}^*)$ is a saddle point for the game.

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