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Analytic Geometry

Microlocal versal deformations of the plane curves $y^k = x^n$ ☆

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Abstract

We introduce the notion of microlocal versal deformation of a plane curve. We construct equisingular versal deformations of Legendrian curves that are the conormal of a semi-quasi-homogeneous branch. ***To cite this article: J. Cabral, O. Neto, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Déformations verselles microlocales des courbes planes $y^k = x^n$. On introduit la notion de déformation verselle microlocale d'un germe de courbe plane. Nous construisons la déformation verselle équisingulière du conormal d'un germe de courbe plane irréductible semi-quasi-homogène. ***Pour citer cet article : J. Cabral, O. Neto, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Version française abrégée

Schlessinger, Tyurina et Grauert ont étudié les déformations verselles des espaces analytiques (cf. [2]). Dans cette Note nous introduisons la notion de déformation verselle microlocale d'une courbe plane. Nous construisons les déformations verselles équisingulières des conormaux des courbes planes irréductibles semi-quasi-homogènes. Nous utilisons les notations et la terminologie de [2]. Soit L une courbe legendrienne de $\mathbb{P}^*\mathbb{C}^2$. Soit φ une transformation de contact telle que le cône tangent de $\varphi(L)$ est transversale aux fibres de la projection $\pi : \mathbb{P}^*\mathbb{C}^2 \to \mathbb{C}^2$. Le type topologique de $\pi(\varphi(L))$ ne dépend pas du choix de φ. Nous appelons *projection plane générique de L* de type topologique de $\pi(\varphi(L))$. On dit que deux courbes legendriennes sont équisingulières si elles ont la même projection plane générique. Une déformation d'une courbe legendrienne est appelée *équisingulière* si toutes ces fibres sont équisingulières.

Soit X une tri-variété de contact et T une variété lisse. On appelle $X \times T$ *tri-variété de contact relative*. On définit de façon similaire *courbe legendrienne relative* et *transformation de contact relative*. Deux courbes legendriennes relatives de même espace de base sont *isomorphes* si l'une est l'image de l'autre par une transformation de contact relative. Le cône tangent de la projection d'un germe de courbe legendrienne est irréductible. Soit Y un germe de

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courbe plane dont le cône tangent est irréductible. Soit $\mathcal{Y}$ une déformation de Y sur T. Le *conormal* de $\mathcal{Y}$ est la plus petite courbe legendrienne relative $\mathcal{L}$ de $\mathbb{P}^*\mathbb{C}^2 \times T$ tel que $\pi_T(\mathcal{L}) = \mathcal{Y}$, où $\pi_T = \pi \times \mathrm{id}_T$. La déformation $\mathcal{Y}$ est une *déformation microlocale* de Y si son conormal est une déformation du conormal de Y. Deux déformations microlocales de Y seront dites *microlocalement équivalentes* si leurs conormaux sont isomorphes.

Définition 0.1. Soit Y un germe de courbe plane avec cône tangent irréductible. Soit L le conormal de Y. On dit que une déformation microlocale [équisingulière] $\mathcal{Y}$ de Y sur T est une *déformation verselle microlocale* [*équisingulière*] *de* Y si les conditions suivantes sont vérifiées :

(i) pour toute déformation microlocale [équisingulière] $\mathcal{Z}$ de Y avec espace de base (R, o) il existe une application holomorphe $\psi : R \to T$ telle que $\psi^*\mathcal{Y}$ est microlocalement équivalente à $\mathcal{Z}$ par une transformation de contact relative φ ;
(ii) la dérivée de ψ au point o ne dépend pas du choix de φ.

Théorème 0.2. *Soit* $C = \{(i, j){:}\ ik + nj > kn,\ j \leqslant k - 2,\ i + j \leqslant n - 2\}$. *Si* $n > 2k$, *la fonction* $G(x, y, t) = y^k - x^n + \sum_{(i,j)\in C} t_{i,j}\, x^i y^j$ *définit une déformation verselle microlocale équisingulière* $\mathcal{G}$ *de la courbe plane* $\{y^k = x^n\}$ *sur* $\mathbb{C}^C = \{(t_{i,j})_{(i,j)\in C}{:}\ t_{i,j} \in \mathbb{C}\}$.

De plus, le conormal de $\mathcal{G}$ est une déformation verselle équisingulière du conormal de $\{y^k = x^n\}$.

1. Contact geometry

Schlessinger, Tyurina and Grauert initiated the study of versal deformations of analytic spaces (cf. [2]). We show in Remark 2 that the obvious definition of deformation of a Legendrian curve is not very interesting. There would be too many rigid Legendrian curves. We consider an alternative approach, recovering Lie's original point of view: to look at contact transformations as maps that take plane curves into plane curves.

We follow the terminology of [2] regarding deformations of analytic spaces. When we refer to a curve or a deformation, we are identifying it with its germ at a convenient point. If $Y \hookrightarrow \mathcal{Y} \to T$ is a deformation $\mathcal{Y}$ of a plane curve Y over a complex manifold T, we assume that Y is reduced. We say that a deformation of a plane curve is *equisingular* if all of its fibers have the same topological type. Given a germ of an analytic set Y we will denote by $\operatorname{mult} Y$ the multiplicity of Y.

We set $\mathbb{C}^A = \{(t_a)_{a\in A}{:}\ t_a \in \mathbb{C}\}$, $t_A = (t_a)_{a\in A}$, for each finite set A. In general, we denote $\partial f/\partial t$ by $\partial_t f$.

Let $(X, \mathcal{O}_X)$ be a complex manifold. A differential form of degree 1 is called a *contact form* if $\omega \wedge d\omega$ never vanishes. If ω is a contact form there is a system of local coordinates (x, y, p) such that $\omega = dy - p\,dx$. A locally free $\mathcal{O}_X$-module Ω of rank 1 is called a *contact structure* if Ω is locally generated by a contact form. The pair (X, Ω) is called a *contact threefold*. Let $\xi\,dx + \eta\,dy$ denote the canonical 1-form of $T^*\mathbb{C}^2 = \mathbb{C}^2 \times \mathbb{C}^2$. We will identify the open set $\{\eta \neq 0\}$ of $\mathbb{P}^*\mathbb{C}^2 = \mathbb{C}^2 \times \mathbb{P}^1$ with the vector space $\mathbb{C}^3$, endowed with the coordinates (x, y, p), where $p = -\xi/\eta$. The contact structure Ω of $\mathbb{C}^3$ generated by the differential form $\omega = dy - p\,dx$ is the restriction to $\mathbb{C}^3$ of the canonical contact structure of $\mathbb{P}^*\mathbb{C}^2$. Let $\pi : \mathbb{P}^*\mathbb{C}^2 \to \mathbb{C}^2$ be the canonical projection. Let $C(Z)$ be the tangent cone of a germ of an analytic set Z. Let $\sigma \in \mathbb{P}^*\mathbb{C}^2$. A Legendrian curve $L \subset (\mathbb{P}^*\mathbb{C}^2, \sigma)$ is in *strong generic position* if $C(L) \cap (D\pi(\sigma))^{-1}(0) = \{0\}$.

Remark 1.

(i) Let Y_1, Y_2 be germs of irreducible plane curves at o. Let (L_i, σ_i) be the conormal of Y_i, $i = 1, 2$. Then $\sigma_1 = \sigma_2$ if and only if Y_1 and Y_2 have the same tangent cone. Hence the plane curve $\pi(L)$ has irreducible tangent cone for each Legendrian curve $L \subset \mathbb{P}^*\mathbb{C}^2$. We will assume that all plane curves have irreducible tangent cone.
(ii) If $Y \hookrightarrow \mathcal{Y} \to T$ is an equisingular deformation of a plane curve Y, $Y \hookrightarrow \mathcal{Y} \to T$ is isomorphic to a deformation $Y \hookrightarrow \mathcal{Z} \to T$ such that the tangent cone of $\mathcal{Z}_t$ does not depend on t. We will assume that all the fibers of an equisingular deformation of a plane curve have the same tangent cone.

(iii) Let Y be a germ of an irreducible plane curve with tangent cone $\{y=0\}$. Let $y=\sum_{\alpha\in\mathbb{Q}} a_\alpha x^\alpha$ be the Puiseux expansion of Y. Set $\delta=\inf\{\alpha\colon\ a_\alpha\neq 0\}$. The tangent cone of the conormal of Y equals $\{x=y=0\}$ if $\delta<2$, $\{y=p-2a_2x=0\}$ if $\delta=2$ and $\{y=p=0\}$ if $\delta>2$.

Let (X,Ω_X) be a contact threefold. Let T be a complex manifold. Let $q_X: X\times T\to X$, $r_X: X\times T\to T$ be the canonical projections. The pair $(X\times T, q_X^*\Omega_X)$ is called a *relative contact threefold*. Let $\mathcal{L}$ be an analytic set of dimension $\dim T+1$ of $X\times T$. We say that $\mathcal{L}$ is a *relative Legendrian curve* if $q_X^*\omega$ vanishes on the regular part of $\mathcal{L}$, for each section ω of Ω_X. The intersection of $\mathcal{L}$ with each fiber of r_X is a Legendrian curve. Given two relative contact threefolds $X\times T$ and $Y\times T$, a biholomorphic map $\Phi: X\times T\to Y\times T$ is called a *relative contact transformation* if $r_Y\circ\Phi=r_X$ and $\Phi^*q_Y^*\Omega_Y=q_X^*\Omega_X$. Given $t\in T$, let Φ_t be the induced map from $X\times\{t\}$ into $Y\times\{t\}$. Two relative Legendrian curves $\mathcal{L}_1\subset X\times T$, $\mathcal{L}_2\subset Y\times T$ are *isomorphic* if there is a relative contact transformation $\Phi: X\times T\to Y\times T$ such that $\Phi(\mathcal{L}_1)=\mathcal{L}_2$. Let $Y\hookrightarrow\mathcal{Y}\to T$ be a deformation of a plane curve. Let $\pi_T:\mathbb{P}^*\mathbb{C}^2\times T\to\mathbb{C}^2\times T$ be the canonical projection. The *conormal* of $\mathcal{Y}$ is the smallest relative Legendrian curve $\mathcal{L}\subset\mathbb{P}^*\mathbb{C}^2\times T$ such that $\pi_T(\mathcal{L})=\mathcal{Y}$.

Let $\mathcal{H}$ be the group of contact transformations of the type $(x,y,p)\mapsto(\lambda x,\mu y,\mu\lambda^{-1}p)$, $\lambda,\mu\in\mathbb{C}^*$. Set $\ell=\{y=p=0\}$. Let $\mathcal{G}$ be the group of germs of contact transformations φ: $(\mathbb{C}^3,0)\to(\mathbb{C}^3,0)$ such that $D\varphi(0)(\ell)=\ell$. Let $\mathcal{J}$ be the group of germs of contact transformations

$$(x,y,p)\mapsto(x+\alpha,y+\beta,p+\gamma),\quad \text{where } \alpha,\beta,\gamma,\partial_x\alpha,\partial_y\beta,\partial_p\gamma\in(x,y,p). \tag{1}$$

Theorem 1.1. *(Cf. [1].) The group $\mathcal{J}$ is an invariant subgroup of $\mathcal{G}$. Moreover, $\mathcal{G}/\mathcal{J}$ is isomorphic to $\mathcal{H}$.*

Theorem 1.2. *(Cf. [1].) Set $t=(t_1,\dots,t_m)$. Let $\alpha\in\mathbb{C}\{x,y,p,t\}$ and $\beta_0\in\mathbb{C}\{x,y,t\}$ be power series such that $\alpha,\partial_x\alpha,\beta_0,\partial_y\beta_0\in(x,y,p)$. There are $\beta,\gamma\in\mathbb{C}\{x,y,p,t\}$ such that $\beta-\beta_0\in(p)$, $\gamma\in(x,y,p)$ and* (1) *is a relative contact transformation. Moreover, β is the solution of the Cauchy problem*

$$\left(1+\frac{\partial\alpha}{\partial x}+p\frac{\partial\alpha}{\partial y}\right)\frac{\partial\beta}{\partial p}-p\frac{\partial\alpha}{\partial p}\frac{\partial\beta}{\partial y}-\frac{\partial\alpha}{\partial p}\frac{\partial\beta}{\partial x}=p\frac{\partial\alpha}{\partial p},\quad \beta-\beta_0\in(p). \tag{2}$$

We set $\alpha=\sum_{l\geqslant 0}\alpha_l p^l$ and $\beta=\sum_{l\geqslant 0}\beta_l p^l$, where $\alpha_l,\beta_l\in\mathbb{C}\{x,y,t\}$ for all l.

Theorem 1.3. *Let L_1, L_2 be two germs of Legendrian curves of $\mathbb{P}^*\mathbb{C}^2$ in strong generic position. Set $Y_i=\pi(L_i)$, $i=1,2$. If there is a contact transformation φ such that $\varphi(L_1)=L_2$, there is a local homeomorphism $\psi:(\mathbb{C}^2,0)\to(\mathbb{C}^2,0)$ such that $\psi(Y_1)=Y_2$.*

Proof. Assume that L_1 is irreducible. Assume that $(n_i,k_i)=1$, $1\leqslant i\leqslant l$, and $n_1/k_1<\dots<n_l/k_l$. Set $k_0=1$, $k=k_l$. The curve Y_1 has Puiseux pairs n_i/k_i, $1\leqslant i\leqslant l$, if and only if the following conditions hold for $1\leqslant i\leqslant l$: (i) $a_{n_ik/k_i}\neq 0$; (ii) if $a_j\neq 0$ and k/k_{i-1} does not divide j, $j\geqslant n_ik/k_i$.

Let $i\in\{1,2\}$. Following the notations of Remark 1(iii), we can assume that $C(Y_i)=\{y=0\}$, $\delta>2$ and the tangent cone of L_i equals $\{y=p=0\}$. By Theorem 1.1, we can assume that φ is of the type (1). Composing φ with the contact transformation induced by $(x,y)\mapsto(x,y-\beta_0)$, we can assume that $\beta_0=0$. Hence $\partial_x\beta,\partial_y\beta\in(p)$. Setting $p=0$ in (2), we conclude that $\beta_1=0$. Let,

$$x=s^k,\qquad y=s^n+\sum_{r\geqslant n+1}a_rs^r,\qquad p=\frac{n}{k}s^{n-k}+\sum_{r\geqslant n+1}\frac{r}{k}a_rs^{r-k}, \tag{3}$$

be a parametrization of L_1. The curve Y_2 admits a parametrization of the type

$$x=s^k+\alpha(s),\qquad y=s^n+\sum_{r\geqslant n+1}a_rs^r+\sum_{l\geqslant 2}\beta_l(s)p(s)^l=s^k+\sum_{r\geqslant n+1}b_rs^r,$$

where $\alpha\in(s^{k+1})$. Since $2(n_i-k_i)>n_i$, the coefficients of $\beta_l(s)p(s)^l$, $l\geqslant 2$, verify condition (ii). Hence the coefficients b_r verify conditions (i), (ii). Replacing the parameter s by a parameter t such that $t^k=s^k+\alpha(s)$ one obtains a parametrization of Y_2 that still verifies conditions (i) and (ii) (cf. Lemma 3.5.4 of [6]).

If L_1' and L_1'' are irreducible Legendrian curves, a similar argument shows that the contact order (cf. Section 4.1 of [6]) of $\pi(\varphi(L_1'))$ and $\pi(\varphi(L_1''))$ equals the contact order of $\pi(L_1')$ and $\pi(L_1'')$. □

Definition 1.4. Let $\mathcal{L}$ be a relative Legendrian curve of a relative contact threefold $X \times (T, o)$. We call *plane projection* of $\mathcal{L}$ a projection to an analytic set $\pi_T(\Phi(\mathcal{L}))$, where $\Phi : X \times T \to \mathbb{P}^*\mathbb{C}^2 \times T$ is a relative contact transformation. The plane projection $\pi_T(\Phi(\mathcal{L}))$ is called *generic* if $\Phi_o(\mathcal{L}_o)$ is in strong generic position. Two Legendrian curves are *equisingular* if their generic plane projections have the same topological type.

Lemma 1.5. *If Y is a plane projection of a Legendrian curve L,* $\operatorname{mult} Y \geqslant \operatorname{mult} L$. *Moreover,* $\operatorname{mult} Y = \operatorname{mult} L$ *if and only if Y is a generic plane projection of L. Hence the multiplicity is an equisingularity invariant of a Legendrian curve.*

Proof. It is enough to prove the result when L is irreducible. Let $\varphi : X \to \mathbb{P}^*\mathbb{C}^2$ be a contact transformation. Let (3) be a parametrization of $\varphi(L)$. Then $\operatorname{mult} Y = k$. Moreover, $\operatorname{mult} L = k$ if $n \geqslant 2k$ and $\operatorname{mult} L = n - k$ if $n < 2k$. The result follows from Remark 1(iii). □

2. Microlocal versal deformations

Let L be a Legendrian curve. A relative Legendrian curve $\mathcal{L} \subset X \times (T, o)$ is called a *deformation* of L if the sets $\mathcal{L}_o$ and L are equal. The deformation $\mathcal{L}$ is called *equisingular* if its fibers are equisingular. We do not demand the flatness of the morphism $\mathcal{L} \hookrightarrow X \times T \to T$. Let Y be a plane curve with conormal L. A deformation $\mathcal{Y}$ of Y is called a *microlocal deformation* of Y if the conormal of $\mathcal{Y}$ is a deformation of L.

Let $\mathcal{J}_T'$ be the group of relative contact transformations $\Phi : (\mathbb{C}^3, 0) \times (T, o) \to (\mathbb{C}^3, 0) \times (T, o)$ such that $\Phi_o = \mathrm{id}_{\mathbb{C}^3}$. Let m be the maximal ideal of $\mathcal{O}_{T,o}$. Given $\Phi \in \mathcal{J}_T'$, there are $\alpha, \beta, \gamma \in m$ such that Φ equals (1). Two deformations $\mathcal{L}_1$, $\mathcal{L}_2$ of L over (T, o) are *isomorphic* if there is $\Phi \in \mathcal{J}_T'$ such that $\Phi(\mathcal{L}_1) = \mathcal{L}_2$. Two microlocal deformations of a plane curve are *microlocally equivalent* if their conormals are isomorphic.

Remark 2. Let $\mathcal{L}$ be a flat equisingular deformation of the conormal of $\{y^k = x^n\}$ along (T, o), where $n > k > 1$ and $(k, n) = 1$. Since $\mathcal{L}$ is flat and $ny - kxp$ vanishes on $\mathcal{L}_o$, there is an holomorphic function f on $X \times T$ such that f vanishes on $\mathcal{L}$ and $f(x, y, p, o) = ny - kxp$. Hence, if $t \in T$ is close enough to o, $\mathcal{L}_t$ is contained in a smooth hypersurface. By Theorem 8.3 of [5], there are integers l, m and a system of local coordinates (x', y', p') on X such that $m > l > 1$, $(l, m) = 1$ and $\mathcal{L}_t$ is the conormal of $\{y'^l = x'^m\}$. Since the deformation is equisingular, $l = k$ and $m = n$. Hence $\mathcal{L}_t$ is isomorphic to $\mathcal{L}_o$ when t is close enough to o. Modifying the proof of the theorem referred above one can show that the deformation $\mathcal{L}$ is trivial even without the assumption that $\mathcal{L}$ is equisingular.

Theorem 2.1. *Let $\mathcal{L}$ be an equisingular deformation of a Legendrian curve L over (T, o). A generic plane projection of $\mathcal{L}$ is an equisingular deformation of $\pi(L)$.*

Proof. Let $\Phi : X \times T \to \mathbb{P}^*\mathbb{C}^2 \times T$ be relative contact transformation such that $\Phi_o(\mathcal{L}_o)$ is in strong generic position. It follows from Lemma 1.5 and the upper-semicontinuity of the map $t \mapsto \operatorname{mult} \pi(\Phi_t(\mathcal{L}_t))$ that $\Phi_t(\mathcal{L}_t)$ is in strong generic position for each t in a neighborhood W of o. Hence the topological type of $\pi(\Phi_t(\mathcal{L}_t))$ equals the topological type of $\pi(\Phi_o(\mathcal{L}_o))$ for $t \in W$. Moreover, the multiplicity of the reduced curve $\pi(\Phi_t(\mathcal{L}_t))$ is constant near o. Therefore $\pi_T(\Phi(\mathcal{L}))_o$ is reduced. □

Definition 2.2. Let Y be a plane curve with an irreducible tangent cone. Let L be the conormal of Y. We say that an [equisingular] microlocal deformation $\mathcal{Y}$ of Y over T is an [*equisingular*] *microlocal versal deformation of* Y if (i) for each [equisingular] microlocal deformation $Y \hookrightarrow \mathcal{Z} \to (R, o)$, there is $\psi : R \to T$ such that $\psi^*\mathcal{Y}$ is microlocally equivalent to $\mathcal{Z}$; (ii) the derivative of ψ at o only depends on $\mathcal{Z}$.

Definition 2.3. An [equisingular] deformation $\mathcal{L}$ of a Legendrian curve L over T is an [*equisingular*] *versal deformation of* L if (i) for each [equisingular] deformation $\mathcal{K}$ of L over (R, o) there is $\psi : R \to T$ such that $\psi^*\mathcal{L}$ is isomorphic to $\mathcal{K}$; (ii) the derivative of ψ at o only depends on $\mathcal{K}$.

Let k, n be integers such that $n > k > 1$ and $(k, n) = 1$. Set $Y = \{y^k - x^n = 0\}$. Let c be the conductor of the semi-group of Y. Let $\mathcal{L}$ be an equisingular deformation of the conormal L of Y.

Remark 3. An equisingular deformation $\mathcal{Y}$ of Y admits a parametrization of the type $x = s^k$, $y = s^n + \sum_{r \geqslant n+1} a_r s^r$, where $a_r \in \mathcal{O}_{T,o}$ and $a_r(o) = 0$. Hence the conormal $\mathcal{K}$ of $\mathcal{Y}$ admits a parametrization $\sigma : (\mathbb{C}, 0) \times (T, o) \to \mathcal{K}$ of type (3). For each t, $s \mapsto \sigma(s, t)$ defines a parametrization of $\mathcal{K}_t$, hence $\mathcal{K}_t$ is the conormal of $\mathcal{Y}_t$. Therefore $\mathcal{Y}$ is a microlocal deformation of Y. The parametrization σ defines a valuation w of the ring $\mathbb{C}[[x, y, p, t]]$, the one that associates to $f \in \mathbb{C}[[x, y, p, t]]$ the multiplicity of the zero of $f \circ \sigma$ as an element of $\mathbb{C}[[t]][t^{-1}][[s]]$.

Lemma 2.4. *Let i, j, l be nonnegative integers such that $ki + nj + (n - k)l > kn$. There are nonnegative integers a, b such that $w(x^i y^j p^l) = w(x^a y^b)$, $x^i y^j p^l \equiv (n/k)^l x^a y^b \bmod(t) + I_{\mathcal{L}}$ and $w(x^i y^j p^l - (n/k)^l x^a y^b) > w(x^a y^b)$. Moreover, $p^k \equiv (n/k)^k x^{n-k} \bmod(t) + I_{\mathcal{L}}$.*

Proof. By Remark 3, $xp \equiv (n/k)y$, $x^{i+1} y^j p^{l+1} \equiv (n/k) x^i y^{j+1} p^l$ and $p^k \equiv (n/k)^k x^{n-k} \bmod(t) + I_{\mathcal{L}}$. Hence we can assume that $l < k$. If $i = 0$, then $j + l > k$ and $y^j p^l \equiv (n/k)^l x^{n-l} y^{j+l-k} \bmod(t) + I_{\mathcal{L}}$. □

Given an analytic set Z of (T, o), let $I_Z[\widehat{I}_Z]$ be the ideal of elements of $\mathcal{O}_{T,o}[\widehat{\mathcal{O}}_{T,o}]$ that vanish on Z.

Lemma 2.5. *Given $u \in \mathbb{C}\{x, y, p, t\}$ such that $w(u) \geqslant c$, there is $v \in \mathbb{C}[[x, y, p, t]]$ such that $u - v \in \widehat{I}_{\mathcal{L}}$.*

Proof. There are nonnegative integers a, b and $\xi \in \mathbb{C}\{t\}$ such that $w(u) = w(x^a y^b)$ and $w(u - \xi x^a y^b) > w(u)$. We iterate the procedure, producing in this way $v \in \mathbb{C}[[x, y, t]]$ such that $w(u - v) = \infty$. □

Lemma 2.6. *(Cf. [1].) Given nonnegative integers a, b and $\lambda \in \mathbb{C}$, there are a contact transformation φ of type* (1) *and $\varepsilon \in \mathbb{C}\{x, y, p\}$ such that $\alpha = \lambda y^a p^b$, $\beta = \lambda b y^a p^{b+1}/(b + 1) + \varepsilon$ and $w(\varepsilon) \geqslant w(y^{2a-1} p^{2b+2})$.*

Theorem 2.7. *The function $F(x, y, t) = y^k - x^n + \sum_{(i,j) \in B} t_{i,j} x^i y^j$ defines an equisingular versal deformation $\mathcal{F}$ of $\{y^k - x^n = 0\}$ over $\mathbb{C}^B$, where $B = \{(i, j){:}\ ki + nj > kn,\ i \leqslant n - 2,\ j \leqslant k - 2\}$.*

Proof. It is a corollary of Theorem II.1.16 of [2]. □

Theorem 2.8. *If $n > 2k$, the function $G(x, y, t) = y^k - x^n + \sum_{(i,j) \in C} t_{i,j} x^i y^j$ defines an equisingular microlocal versal deformation $\mathcal{G}$ of $\{y^k - x^n = 0\}$ over $\mathbb{C}^C$, where $C = \{(i, j) \in B{:}\ i + j \leqslant n - 2\}$.*

Proof. We can assume that $R = (\mathbb{C}^a, 0)$ and $\mathcal{O}_{\mathbb{C}^a,0} = \mathbb{C}\{r\}$. Let $H \in \mathbb{C}\{x, y, r\}$ be a generator of the ideal $I_{\mathcal{Z}}$. By Theorem 2.7 we can assume that there are $\xi_{i,j} \in \mathbb{C}\{r\}$, $(i, j) \in B$, such that $\xi_{i,j} \in (r)$ and $H(x, y, r) = y^k - x^n + \sum_{(i,j) \in B} \xi_{i,j} x^i y^j$. Moreover, the functions $\xi_{i,j}$ are uniquely determined $\bmod(r^2)$. Let $\mathcal{K}$ be the conormal of $\mathcal{Z}$. Assume that there are $\psi : \mathbb{C}^a \to \mathbb{C}^C$, $\psi = (\psi^{i,j})$, and $\Phi \in \mathcal{J}'_{\mathbb{C}^a}$ such that $\Phi^{-1}(\mathcal{K})$ equals the conormal of $\psi^* \mathcal{G}$. There are $\alpha, \beta, \gamma \in \mathbb{C}\{x, y, p, r\}$ such that Φ equals (1). Hence there is $\varepsilon \in (r)$ such that $H(x + \alpha, y + \beta, r) = (1 + \varepsilon(x, y, r)) G(x, y, \psi(r)) \bmod I_{\Phi^{-1}(\mathcal{K})}$. Notice that

$$H(x + \alpha, y + \beta, r) \equiv H(x, y, r) + k y^{k-1} \beta - n x^{n-1} \alpha \bmod(r)^2.$$

By Lemma 2.4, $x^{n-1}\alpha$, $y^{k-1}\beta$ are congruent modulo $(r)^2 + \widehat{I}_{\Phi^{-1}(\mathcal{K})}$ with elements of $(x, y)^{n-1}\mathbb{C}[[x, y, r]]$. Hence $\psi^{i,j} \equiv \xi_{i,j} \bmod(r)^2$ for each $(i, j) \in C$.

By Theorem 2.7, it is enough to show that there is $\psi : \mathbb{C}^B \to \mathbb{C}^C$ such that $\psi^* \mathcal{G}$ is microlocally equivalent to $\mathcal{F}$. Set $N = \#(B \setminus C)$. Let us order the pairs $(i, j) \in B \setminus C$ by the value of $ki + nj$. Let B_l be the set B minus the set of the l smaller ordered pairs of $B \setminus C$. Set $H_l(x, y, t_{B_l}) = y^k - x^n + \sum_{(i,j) \in B_l} t_{i,j} x^i y^j$. Let $\mathcal{L}_l$ be the conormal of $\mathcal{H}_l = \{H_l = 0\}$. Let $0 \leqslant l \leqslant N - 1$. It is enough to show that there is $\psi_l : \mathbb{C}^{B_l} \to \mathbb{C}^{B_{l+1}}$ such that $\psi_l^* \mathcal{H}_{l+1}$ is microlocally equivalent to $\mathcal{H}_l$. Let (a, b) be the smallest element of $B_l \setminus C$. Let $v = w_l(x^a y^b)$. Let Φ_l be the contact transformation of the type (1) associated by Theorem 1.2 to $\alpha = \lambda y^{a+b-(n-1)} p^{n-1-a}$ and $\beta_0 = 0$. Let w_l be the valuation associated to $\Phi_l^{-1}(\mathcal{L}_l)$. By Lemma 2.6, $\beta = \lambda(n-1-a)y^{a+b-(n-1)}p^{n-a}/(n-a) + \mu$, with $w_l(\mu) > w_l(y^{a+b-(n-1)}p^{n-a})$. There is $\delta \in \mathbb{C}\{x, y, p, t_{B_l}\}$ such that $w_l(\delta) > v$ and

$$H_l(x+\alpha, y+\beta, t_{B_l}) \equiv H_{l+1}(x, y, t_{B_{l+1}}) + t_{a,b}x^a y^b + ky^{k-1}\beta - nx^{n-1}\alpha + \delta \bmod I_{\Phi_l^{-1}(\mathcal{L}_l)}.$$

Set $\lambda = t_{a,b}(n-a)(n/k)^{n-1-a}/n$. By Lemma 2.5, there is $\widehat{\delta} \in \mathbb{C}[[x, y, t_{B_l}]]$ such that $w_l(\widehat{\delta}) > v$ and $H_l(x+\alpha, y+\beta, t_{B_l}) \equiv H_{l+1}(x, y, t_{B_{l+1}}) + \widehat{\delta} \bmod \widehat{I}_{\Phi_l^{-1}(\mathcal{L}_l)}$. Since $\pi_{B_l}(\Phi_l^{-1}(\mathcal{L}_l))$ is a convergent hypersurface, we can assume that $\widehat{\delta} \in \mathbb{C}\{x, y, t_{B_l}\}$. By the proof of Theorem 2.7, there are $\alpha, \beta, \varepsilon \in \mathbb{C}\{x, y\}[[t_{B_l}]]$ and $\psi^{i,j} \in \mathbb{C}[[t_{B_l}]]$, $(i, j) \in B$, such that $\alpha, \beta, \varepsilon \in (t_{B_l})$, and

$$(H_{l+1} + \widehat{\delta})(x+\alpha, y+\beta, t_{B_l}) = (1+\varepsilon)F\big(x, y, \psi_l(t_{B_l})\big),$$

where $\psi_l = (\psi^{i,j})$. Since $y^k - x^n$ is quasi-homogeneous, we can assume that ε vanishes. Set $\alpha = \sum_q \alpha_q$, $\beta = \sum_q \beta_q$, $\psi^{i,j} = \sum_q \psi_q^{i,j}$, where $\alpha_q, \beta_q, \psi_q^{i,j}$ are homogeneous functions of degree q on the variables $t_{i,j}$, $(i, j) \in B_l$. For each q there is an homogeneous part g_q of degree q of an element of the ideal $(\partial_x H_{l+1}, \partial_y H_{l+1})$ such that $\alpha_q nx^{n-1} + \beta_q ky^{k-1} = g_q$. Moreover, each g_q depends on the choices of α_p, β_p, $p < q$. Notice that $\alpha_0, \beta_0, g_0 = 0$, g_1 is the linear part of $\widehat{\delta}$, $\psi_1^{i,j} = t_{i,j}$ if $(i, j) \in B_{l+1}$ and $\psi_1^{i,j} = 0$ otherwise. We choose α_1, β_1 such that $\beta_1 ky^{k-1}$ is the rest of the division of g_1 by nx^{n-1}. Since $w_l(g_1) > v$, then $w_l(\alpha_1 x^{n-1}), w_l(\beta_1 y^{k-1}), w_l(g_2) > v$ and $\psi_2^{i,j} = 0$ if $ki + nj \leqslant v$. Moreover, we can iterate the procedure. We show in this way that $F(x, y, \psi_l(t_{B_l})) = H_{l+1}(x, y, \psi_l(t_{B_l}))$. Since our choices of α_q, β_q are the ones of Lemma 1 of [4] and of Theorem II.D.2 of [3], the functions $\alpha, \beta, \psi^{i,j}$, $(i, j) \in B$, converge. □

Corollary 2.9. *Each irreducible Legendrian curve L contained in a smooth surface admits an equisingular versal deformation. This deformation is trivial if and only if L is isomorphic to the conormal of a curve defined by one of the functions* $y^2 - x^{2n+1}$, $n \geqslant 1$, $y^3 - x^7$, $y^3 - x^8$.

Proof. By Theorem 8.3 of [5], there are integers k, n such that $n > 2k > 1$, $(k, n) = 1$ and the Legendrian curve is isomorphic to the conormal of $\{y^k = x^n\}$. The corollary follows from Remark 3, Theorems 2.1 and 2.8. □

By the canonical properties of versal deformations (cf. Theorem II.1.15 of [2]), the previous result still holds if L is the conormal of a semi-quasi-homogeneous branch.

Example 1. Set $f_0(x, y) = y^4 - x^{11}$, $f_1(x, y) = y^4 - x^{11} + x^6y^2$, $f_2(x, y) = y^4 - x^{11} + x^7y^2$. Let L_i be the conormal of $\{f_i = 0\}$, $0 \leqslant i \leqslant 2$. A Legendrian curve equisingular to L_0 is isomorphic to one and only one of the curves L_0, L_1, L_2. The equisingular microlocal versal deformation of $y^4 - x^{11}$ equals $G(x, y, t_2, t_6) = y^4 - x^{11} + t_2x^6y^2 + t_6x^7y^2$. Moreover,

$$G\big(\lambda^4 x, \lambda^{11} y, t_2, t_6\big) = \lambda^{44} G\big(x, y, \lambda^2 t_2, \lambda^6 t_6\big), \quad \text{for each } \lambda \in \mathbb{C}^*. \tag{4}$$

If $t_2 \neq 0$, it follows from (4) that we can assume $t_2 = 1$. Moreover, there are $\varepsilon, \delta \in (x, y)$ such that $G(x - 2t_6x^2, y - 11t_6xy/2, 1, t_6) = (1 - 22t_6x + \varepsilon)(y^4 - x^{11} + x^6y^2 + \delta)$ and $w(\delta) \geqslant 52$. By Theorem 2.8 we can assume that $\delta = 0$. If $t_2 = 0$ it follows from (4) that we can assume $t_6 = 0$ or $t_6 = 1$.

Since L_0 is contained in the surface $11y - 4xp = 0$ and L_1, L_2 are not contained in a smooth surface, L_0 cannot be isomorphic to L_1 or L_2. Let $\varphi \in \mathcal{J}$. There is $\varepsilon \in \mathbb{C}\{x, y\}$ such that $f_1(x+\alpha, y+\beta) \equiv f_1(x, y) + \varepsilon(x, y) \bmod I_{\varphi(L_1)}$. If $f_1 + \varepsilon = f_2$, $w(\varepsilon) \geqslant 46$. Hence $w(\varepsilon) \geqslant 47$. Therefore $f_1 + \varepsilon \neq f_2$. By Theorem 1.1, $\varphi(L_1) \neq L_2$ for each $\varphi \in \mathcal{G}$.

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