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Mathematical Analysis

A mapping connected with the Schur–Szegő composition

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Abstract

Every monic polynomial in one variable of the form $(x+1)S$, $\deg S = n-1$, is presentable in a unique way as a Schur–Szegő composition of $n-1$ polynomials of the form $(x+1)^{n-1}(x+a_i)$. We prove geometric properties of the affine mapping associating to the coefficients of S the $(n-1)$-tuple of values of the elementary symmetric functions of the numbers a_i. ***To cite this article: V.P. Kostov, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Une application liée à la composition de Schur–Szegő. Tout polynôme unitaire à une variable de la forme $(x+1)S$, $\deg S = n-1$, est présentable de façon unique comme composition de Schur–Szegő de $n-1$ polynômes $(x+1)^{n-1}(x+a_i)$. Nous prouvons des propriétés géométriques de l'application affine associant aux coefficients de S le $(n-1)$-uplet des valeurs des fonctions symétriques élémentaires des nombres a_i. ***Pour citer cet article : V.P. Kostov, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Version française abrégée

Pour le couple de polynômes (réels ou complexes) de degré n à une variable $P = \sum_{j=0}^{n} p_j x^j$, $Q = \sum_{j=0}^{n} q_j x^j$ on définit leur *composition de Schur–Szegő* (*CSS*) par la formule $P *_n Q = \sum_{j=0}^{n} (p_j q_j / C_n^j) x^j$. Si on considère P et Q comme des polynômes de degré $n+k$ à k premiers coefficients nuls la formule devient $P *_{n+k} Q = \sum_{j=0}^{n} (p_j q_j / C_{n+k}^j) x^j$. La CSS est commutative et associative (voir [5] pour plus de détails sur la CSS). Dans l'article [1] on prouve que chaque polynôme P de degré n et tel que $P(-1)=0$, est présentable sous la forme

$$P = K_{a_1} *_n \cdots *_n K_{a_{n-1}} \tag{K}$$

où $K_{a_i} = (x+1)^{n-1}(x+a_i)$ et les nombres $a_j \in \mathbf{C}$ sont uniques à permutation près (par convention $K_\infty = (x+1)^{n-1}$). Leur unicité implique que si P est réel, alors une partie des a_i sont réels et les autres forment des couples conjugués (sinon la conjugaison des deux côtés de (K) définirait un autre $(n-1)$-uplet de nombres a_j). Dans ce qui suit nous posons $P = (x+1)S$, $S = x^{n-1} + c_1 x^{n-2} + \cdots + c_{n-1}$.

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Remarque 1. (1) Posons $b_i := -i/(n-i)$. Si $P^{(i)}(0) = 0$, alors un des nombres a_j vaut b_i. Si $S = x^k W$, $\deg W = n-k-1$, $k \leqslant n-1$, alors à permutation près $a_j = b_{j-1}$, $j = 1, \ldots, k$ (voir [2, Remark 7]).

(2) Si P a r racines réelles > 0, alors au-moins r des nombres a_j sont réels < 0, distincts et appartenant à r intervalles différents de la forme $[b_j, b_{j-1}]$. En effet, d'après la règle de Descartes il y a au-moins r changements de signe dans la suite des coefficients de P. Un polynôme K_a avec $a > 0$ ou la CSS de deux polynômes K_a, $K_{\bar{a}}$ ou la CSS de deux polynômes K_{a_1}, K_{a_2} avec a_1, a_2 appartenant au même intervalle (b_j, b_{j-1}), a tous ses coefficients positifs. D'après la partie (1) de cette remarque les changements de signe peuvent résulter seulement de polynômes K_{a_i} avec $a_i < 0$ et appartenant à des intervalles différents $[b_j, b_{j-1}]$.

On rappelle qu'un polynôme réel est dit *hyperbolique* si toutes ses racines sont réelles.

Notation 0.1. On désigne par $\Pi_{n-1} \subset \mathbf{R}^{n-1} \cong Oc_1 \cdots c_{n-1} =: \mathcal{R}$ l'ensemble *fermé* dans lequel le polynôme P est hyperbolique. Posons $\sigma_j = \sum_{1 \leqslant i_1 < \cdots < i_j \leqslant n-1} a_{i_1} \cdots a_{i_j}$. On désigne par U_{n-1} (par V_{n-1}) le sous-ensemble *ouvert* de $\mathcal{R}$ dans lequel on a $c_1 < 0, c_2 > 0, \ldots, (-1)^{n-1} c_{n-1} > 0$ (dans lequel les parties réelles de toutes les racines de S sont > 0). Posons $\tilde{c} := (c_1, \ldots, c_{n-1})$, $\tilde{\sigma} := (\sigma_1, \ldots, \sigma_{n-1})$. Nous écrivons $P \in U_{n-1}$ lorsque $\tilde{c} \in U_{n-1}$, etc. On désigne l'adhérence et la frontière de l'ensemble Δ par $\overline{\Delta}$ et $\partial\Delta$. Pour un polynôme A de degré n on pose $A^R(x) = x^n A(1/x)$. On a $(A^R)^R = A$.

Nous étudions des propriétés de l'application affine $\Phi : \tilde{c} \mapsto \tilde{\sigma}$. Posons $\Phi(P) = (x+1)(x^{n-1} + \sigma_1 x^{n-2} + \cdots + \sigma_{n-1}) = (x+1)(x+a_1)\cdots(x+a_{n-1})$. Comme $(-a_i)$ sont les racines de $\Phi(P)/(x+1)$, nous écrivons $\Phi(P) \in V_{n-1}$ lorsque $\operatorname{Re}(-a_i) > 0$, c.-à-d. $\operatorname{Re}(a_i) < 0$.

Lemme 0.2. *On a* $\Phi(P^R) = (\Phi(P))^R$. *Si* $P \in \overline{V_{n-1}}$, *alors* $P^R \in \overline{V_{n-1}}$.

Lemme 0.3. *Pour* $n \geqslant 2$ *on a* $V_{n-1} \subseteq U_{n-1}$ *(donc* $\overline{V_{n-1}} \subseteq \overline{U_{n-1}}$*) avec égalité si et seulement si* $n = 2, 3$.

Remarque 2. L'application affine Φ est non-dégénérée, les valeurs propres de sa linéarisation sont $1 = \lambda_1 < \lambda_2 < \cdots < \lambda_{n-1}$, les vecteurs propres sont des polynômes de degré $n-1$ de la forme $G_1 = (x+1)^{n-1}$, $G_2 = x(x+1)^{n-2}$, $G_3 = x(x+1)^{n-3} Q_{1,n}(x), \ldots, G_{n-1} = x(x+1) Q_{n-3,n}$ où les racines de $Q_{j,n}$ sont > 0 et distinctes, voir [3]. On peut considérer Φ aussi comme un automorphisme linéaire *diagonalisable* de l'espace de polynômes de degré n divisibles par $x+1$. Dans ce cas on ajoute un vecteur propre $G_0 = (x+1)^n$ avec la valeur propre $\lambda_0 = \lambda_1 = 1$.

Théorème 0.4.

(1) *On a* $\Phi(V_{n-1}) \subset V_{n-1}$ *et* $\Phi(\Pi_{n-1} \cap V_{n-1}) \subset (\Pi_{n-1} \cap V_{n-1})$.
(2) *On a* $\Phi(U_{n-1}) \subset U_{n-1}$.
(3) *Si* $A = (c_1^0, \ldots, c_{n-1}^0) \in \partial U_{n-1}$, *alors* $\Phi(A) \in \partial U_{n-1}$ *si et seulement si* $c_{n-1}^0 = 0$.
(4) *Pour tout polynôme réel* $P \neq 0$ *il existe* $h(P) \in \mathbf{N}$ *tel que* $\Phi^k(P) \in \Pi_{n-1}$ *si* $k \geqslant h(P)$.
(5) *Il existe* $\nu \in \mathbf{N}$ *dépendant seulement de* n *tel que pour tout* $P \in \overline{U_{n-1}}$ *on a* $\Phi^\nu(P) \in \Pi_{n-1}$.

1. Results

For the couple of real or complex degree n polynomials in one variable $P = \sum_{j=0}^n p_j x^j$, $Q = \sum_{j=0}^n q_j x^j$ we define their *composition of Schur–Szegő* (CSS) by the formula $P *_n Q = \sum_{j=0}^n (p_j q_j / C_n^j) x^j$. When we consider P and Q as degree $n+k$ polynomials with k first coefficients equal to 0 the formula becomes $P *_{n+k} Q = \sum_{j=0}^n (p_j q_j / C_{n+k}^j) x^j$. The CSS is commutative and associative. Set $K_a = (x+1)^{n-1}(x+a)$, $K_\infty = (x+1)^{n-1}$. Each degree n polynomial P such that $P(-1) = 0$ is presentable in the form

$$P = K_{a_1} *_n \cdots *_n K_{a_{n-1}} \tag{1}$$

(see [1]) where $a_j \in \mathbf{C}$ are unique up to permutation. If P is real, some of the numbers a_j are real, the rest form conjugate couples (otherwise conjugation of both sides of (1) defines a new set of numbers a_j). In what follows we set $P = (x+1)S$, $S = x^{n-1} + c_1 x^{n-2} + \cdots + c_{n-1}$. See more about the CSS in [5].

Remark 1. (1) Set $b_i := -i/(n-i)$, $i = 0, \dots, n-1$, $b_n = \infty$. If $P^{(\nu)}(0) = 0$, then a number a_j equals b_ν. If $S = x^k W$, $\deg W = n-k-1$, $k \leqslant n-1$, then up to permutation $a_j = b_{j-1}$, $j = 1, \dots, k$ (see [2, Remark 7]). Therefore for $j = 0, \dots, n-1$ one has $(x+1)x^j = \alpha_j K_{b_0} {}_n^* \cdots {}_n^* \hat{K}_{b_j} {}_n^* \hat{K}_{b_{j+1}} {}_n^* \cdots {}_n^* K_{b_n}$ where $\alpha_j = (C_n^j)^{n-2} / \prod_{i=0,\dots,j-1,j+2,\dots,n} (C_{n-1}^j b_i + C_{n-1}^{j-1})$. Set $M := \prod_{i=0}^{n-1}(x - b_i)$, $M_j := M/((x-b_j)(x-b_{j+1}))$.

(2) If P has r positive roots, then at least r of the numbers a_j belong to r different intervals $[b_j, b_{j-1}]$ – by the Descartes rule there are $\geqslant r$ sign changes in the sequence of coefficients of P; a polynomial K_a, $a > 0$ or $K_a {}_n^* K_{\bar{a}}$ or $K_{a_1} {}_n^* K_{a_2}$ with a_1, a_2 from the same interval (b_j, b_{j-1}) has all coefficients positive.

Recall that a real polynomial is *hyperbolic* if its roots are all real. Recall also the formulas (see [2,4])

$$P {}_n^* (x+1)^n = P, \tag{A}$$

$$\big(P {}_{n+k}^* Q\big)' = \big(1/(n+k)\big)\big(P' {}_{n+k-1}^* Q'\big), \tag{B}$$

$$\text{if } Q = x^q T, \quad \text{then } P {}_{n+k}^* Q = \big((n+k-q)!/(n+k)!\big) x^q \big(P^{(q)} {}_{n+k-q}^* T\big). \tag{C}$$

Notation 1.1. Denote by $\Pi_{n-1} \subset \mathbf{R}^{n-1} \cong Oc_1 \cdots c_{n-1} =: \mathcal{R}$ the *closed* subset for which P is hyperbolic. Set $\sigma_j = \sum_{1 \leqslant i_1 < \cdots < i_j \leqslant n-1} a_{i_1} \cdots a_{i_j}$. Denote by U_{n-1} (by V_{n-1}) the *open* subsets of $\mathcal{R}$ for which $c_1 < 0$, $c_2 > 0, \dots, (-1)^{n-1} c_{n-1} > 0$ (for which the real parts of all roots of S are > 0). Set $\tilde{c} := (c_1, \dots, c_{n-1})$, $\tilde{\sigma} := (\sigma_1, \dots, \sigma_{n-1})$. Writing $P \in U_{n-1}$ means $\tilde{c} \in U_{n-1}$, etc. Denote the closure (the boundary) of a set Δ by $\overline{\Delta}$ (by $\partial \Delta$) and by A^R the *reverted* of the degree n polynomial A, i.e. $A^R(x) = x^n A(1/x)$; $(A^R)^R = A$.

In this paper we study properties of the affine mapping $\Phi : \tilde{c} \mapsto \tilde{\sigma}$. Set $\Phi(P) = (x+1)(x^{n-1} + \sigma_1 x^{n-1} + \cdots + \sigma_{n-1}) = (x+1)(x+a_1)\cdots(x+a_{n-1})$. By Remark 1, $\Phi(P) = (x+1)\sum_{i=0}^{n-1} c_{n-1-i} \alpha_i M_i$ (**D**), $c_0 = 1$. As $(-a_i)$ are the roots of $\Phi(P)/(x+1)$, writing $\Phi(P) \in V_{n-1}$ means $\operatorname{Re}(-a_i) > 0$, i.e. $\operatorname{Re}(a_i) < 0$.

Lemma 1.2. *If $P \in \overline{V_{n-1}}$, then $P^R \in \overline{V_{n-1}}$. One has $\Phi(P^R) = (\Phi(P))^R$.*

To prove the first statement observe that if x_i are the roots of P ($\operatorname{Re} x_i \geqslant 0$), then the ones of P^R are $1/x_i$ and $\operatorname{Re}(1/x_i) \geqslant 0$. The second statement follows from $(K_a)^R = K_{1/a}$ ($(K_0)^R = K_\infty$).

Lemma 1.3. *For $n \geqslant 2$ one has $V_{n-1} \subseteq U_{n-1}$ (hence $\overline{V_{n-1}} \subseteq \overline{U_{n-1}}$) with equality only for $n = 2$ and 3.*

Proof. Induction on n. One has $V_1 = U_1$, $V_2 = U_2$ (Examples 1 and 2). For $n \geqslant 4$ if $\xi > 0$ is a root of P, then set $S = (x - \xi)(x^{n-2} + d_1 x^{n-3} + \cdots + d_{n-2})$, $\tilde{d} := (d_1, \dots, d_{n-2})$. If $P \in V_{n-1}$, then $\tilde{d} \in V_{n-2}$. By inductive assumption $\tilde{d} \in U_{n-2}$. One has $c_j = d_j - \xi d_{j-1}$ (we set $d_0 = 1$, $d_{n-1} = 0$). As $\operatorname{sign}(d_j) = \operatorname{sign}(-\xi d_{j-1})$, one has $\operatorname{sign}(c_j) = \operatorname{sign}(d_j)$ for $j = 1, \dots, n-2$ and $\operatorname{sign}(c_{n-1}) = -\operatorname{sign}(d_{n-2})$. Hence $\tilde{c} \in U_{n-1}$.

If $\eta \pm i\zeta$ ($\eta > 0$) are roots of S, then set $S = (x^2 - 2\eta x + \eta^2 + \zeta^2)(x^{n-3} + g_1 x^{n-4} + \cdots + g_{n-3})$, where $(g_1, \dots, g_{n-3}) \in V_{n-3} \subseteq U_{n-3}$. As $c_j = g_j - 2\eta g_{j-1} + (\eta^2 + \zeta^2) g_{j-2}$, we conclude as above that $\tilde{c} \in U_{n-1}$. For $n \geqslant 4$ suppose that the roots of S equal $\pm i, 1, \dots, 1$ (hence $P \in U_{n-1}$). For $\varepsilon > 0$ small enough the polynomial S with roots $-\varepsilon \pm i, 1, \dots, 1$ still defines a point in U_{n-1}, therefore $V_{n-1} \neq U_{n-1}$. □

Remark 2. The mapping Φ has eigenvalues $1 = \lambda_1 < \lambda_2 < \cdots < \lambda_{n-1}$, the eigenvectors are $G_1 = (x+1)^{n-1}$, $G_j = x(x+1)^{n-j} Q_{j-2,n}$, $j \geqslant 2$, where $Q_{j-2,n}$ is degree $j-2$, real, with positive and distinct roots, see [3]. When viewed as a *diagonalizable* linear automorphism of the space of degree n polynomials divisible by $x+1$, Φ has one more eigenvector $G_0 = (x+1)^n$ with the eigenvalue $\lambda_0 = \lambda_1 = 1$.

Theorem 1.4.

(1) *One has $\Phi(V_{n-1}) \subset V_{n-1}$ and $\Phi(\Pi_{n-1} \cap V_{n-1}) \subset (\Pi_{n-1} \cap V_{n-1})$.*
(2) *One has $\Phi(U_{n-1}) \subset U_{n-1}$.*
(3) *If $C = (c_1^0, \dots, c_{n-1}^0) \in \partial U_{n-1}$, then $\Phi(C) \in \partial U_{n-1}$ if and only if $c_{n-1}^0 = 0$.*

(4) *For each real polynomial* $P \neq 0$ *there exists* $h(P) \in \mathbf{N}$ *such that* $\Phi^k(P) \in \Pi_{n-1}$ *when* $k \geqslant h(P)$.
(5) *There exists* $\nu \in \mathbf{N}$ *depending only on* n *such that for each* $P \in \overline{U_{n-1}}$ *one has* $\Phi^\nu(P) \in \Pi_{n-1}$.

Remark 3. Part (1) of the theorem is interesting from the point of view of stability theory. In (5) the set $\overline{U_{n-1}}$ cannot be replaced by $\mathbf{R}^{n-1}$ for $n \geqslant 3$ – Φ being nondegenerate, this would imply $\Pi_{n-1} = \mathbf{R}^{n-1}$.

Example 1. For $n = 2$ one has $\Phi = \mathrm{id}$ and all statements of the theorem are evident (one has $P = (x+1)(x-a) = K_{-a}$, i.e. $a_1 = -a$ and P is hyperbolic).

Example 2. For $n = 3$ one has (see [3]) $\Phi : (c_1, c_2) \mapsto ((3c_1 - c_2 - 1)/2, c_2)$ or equivalently $(c_1 - 1, c_2) \mapsto ((3(c_1 - 1) - c_2)/2, c_2)$. Hence $U_2 = V_2 = \{c_1 \leqslant 0 \leqslant c_2\}$, $\Pi_2 \cap U_2 = \{c_1 \leqslant 0, 0 \leqslant c_2 \leqslant c_1^2/4\}$, $\Phi(U_2) = \{0 \leqslant c_2 \leqslant -2c_1 - 1\}$. Thus (1), (2) and (3) are true. One has $\Phi^2(U_2) = \{0 \leqslant c_2 \leqslant (-5 - 4c_1)/5\} \subset (\Pi_2 \cap U_2)$, hence there holds (5) with $\nu = 2$. The mapping Φ has fixed points along the line $c_2 = c_1 - 1$ which define hyperbolic polynomials $(x+1)^2(x+c_1)$. For every other point (c_1^0, c_2^0) the point $\Phi^k(c_1^0, c_2^0)$ defines hyperbolic polynomials for k sufficiently large (the eigenvalue 3/2 makes the module of the first component of $\Phi^k(c_1^0, c_2^0)$ tend to ∞, the second remains fixed). For large k such a quadratic polynomial is hyperbolic. Thus for $n = 3$ one can set $\nu = 2$ (but not $\nu = 1$ – it is not true that $\Phi(U_2) \subset (\Pi_2 \cap U_2)$).

Proof of Theorem 1.4. We prove the theorem by induction on n, the cases $n = 2, 3$ (Examples 1, 2) are the induction base. We prove part (1) in 1^0–3^0, parts (2) and (3) in 4^0–5^0, part (4) in 6^0, and part (5) in 7^0–8^0.

1^0. The second statement of part (1) follows from Remark 1. The sets V_{n-1} and $\Phi(V_{n-1})$ are semi-algebraic, of dimension $n-1$ and simply-connected. If there exists $P \in V_{n-1}$ such that $\Phi(P) \notin V_{n-1}$, then there exists $P_* \in \partial V_{n-1}$ such that $\Phi(P_*) \notin \overline{V_{n-1}}$. Show that this is impossible. Set $\partial V_{n-1} = Y \cup Z$, the polynomials in Y having a root at 0, the ones in Z having a couple of imaginary conjugate roots.

Lemma 1.5. *Suppose that* $P = (x+1)xS_1$, $\deg(S_1) = n-2$. *Then* $a_1 = 0$ *and the decomposition* (1) *of* $(x+1)S_1$ *equals* $\tilde{K}_{a_2} {}_{n-1}^{*} \cdots {}_{n-1}^{*} \tilde{K}_{a_{n-1}}$, *where* $\tilde{K}_{a_i} = (x+1)^{n-2}(x + ((n-1)a_i + 1)/n)$.

Indeed, $a_1 = 0$ follows from part (1) of Remark 1. Apply formulas (A)–(C):

$$\begin{aligned}(x+1)xS_1 &= (x+1)^{n-1}x \,{}_n^{*}\, K_{a_2} \,{}_n^{*} \cdots {}_n^{*}\, K_{a_{n-1}} \\ &= (x/n)\big((x+1)^{n-1} \,{}_{n-1}^{*}\, \big(K_{a_2} \,{}_n^{*} \cdots {}_n^{*}\, K_{a_{n-1}}\big)'\big) \\ &= \big(x/n^{n-2}\big)\big(K'_{a_2} \,{}_{n-1}^{*} \cdots {}_{n-1}^{*}\, K'_{a_{n-1}}\big) = x\big(\tilde{K}_{a_2} \,{}_{n-1}^{*} \cdots {}_{n-1}^{*}\, \tilde{K}_{a_{n-1}}\big).\end{aligned}$$

Hence $\tilde{K}_{a_2} {}_{n-1}^{*} \cdots {}_{n-1}^{*} \tilde{K}_{a_{n-1}}$ is the decomposition (1) of the degree $n-1$ polynomial $(x+1)S_1$.

Suppose that $P \in Y$, i.e. $P = (x+1)xS_1$. The roots of S_1 belong to V_{n-2}, so by inductive assumption, $\mathrm{Re}(((n-1)a_i + 1)/n) \leqslant 0$, hence $\mathrm{Re}(a_i) \leqslant -1/(n-1) < 0$. This means that $\Phi(P) \in Y \subset \overline{V_{n-1}}$.

2^0. Suppose that P has g positive roots, h purely imaginary couples and q conjugate couples with positive real parts all counted with multiplicity, their sets being denoted by G^*, H^*, Q^*. Set $P = GHQ$ where the roots of the monic polynomials G (H, Q) are in G^* (H^*, Q^*).

Case A: $n - 1 = 2s$, $s \in \mathbf{N}$, $g = q = 0$. Hence $P \in Z$, $P = (x+1)R$, $R = \sum_{\nu=0}^{s} r_\nu x^{2\nu}$, $r_\nu > 0$, $\Phi(P)(z) = \sum_{\nu=0}^{s} r_\nu \alpha_{2\nu} M_{2\nu}$, $\alpha_{2\nu} > 0$ (Remark 1 and formula (**D**) after Notation 1.1). Recall that $M_{2\nu} = M/((x - b_{2\nu})(x - b_{2\nu+1}))$. One has $1/((x - b_{2\nu})(x - b_{2\nu+1})) = (b_{2\nu} - b_{2\nu+1})^{-1}((\bar{x} - b_{2\nu})/|x - b_{2\nu}|^2 - (\bar{x} - b_{2\nu+1})/|x - b_{2\nu+1}|^2) = \kappa\bar{x} + \rho$, where $\rho \in \mathbf{R}$ and $\kappa > 0$ because $|x - b_{2\nu+1}| > |x - b_{2\nu}|$, $b_{2\nu} > b_{2\nu+1}$. Hence if $\mathrm{Re}\, x < 0$, $\mathrm{Im}\, x \neq 0$, then $\Phi(P)(x) \neq 0$. The polynomial $\Phi(P)$ has no negative root – for $r_\nu \to 0$, $1 \leqslant \nu \leqslant s$, its roots tend to the numbers $-b_i \geqslant 0$, so it has $\leqslant 1$ negative roots. If it has one, then the root is simple and $\mathrm{sign}\,\Phi(P)(0) = -\mathrm{sign}\,P(0)$ which contradicts formula (1). Hence $\Phi(P) \in V_{n-1}$ for $r_\nu > 0$.

3^0. Suppose that $P \in \overline{V_{n-1}}$ and that $\Phi(P)$ has a root $\zeta \neq 0$, $\mathrm{Re}\,\zeta \leqslant 0$. We show how to continuously deform $P \in \overline{V_{n-1}}$ to decrease $\mathrm{Re}\,\zeta$ and finally get in Case A from where the proof of part (1) follows.

Case B: *$\Phi(P)$ has simple roots $\zeta, \bar{\zeta}$ with* $\mathrm{Re}\,\zeta \leqslant 0$. Explain how to change Q so that q decrease.

Consider the system $\mathrm{Re}\,\Phi(P)(\zeta) = \mathrm{Im}\,\Phi(P)(\zeta) = 0$ as a linear system with unknown variables the coefficients of $\Phi(P)$ (or equivalently, of P). It is rank 2 (it is equivalent to $\Phi(P)(\zeta) = \Phi(P)(\bar{\zeta}) = 0$). Consider the linear forms (in the coefficients of P) $L_{u,v} := u\,\mathrm{Re}\,\Phi(P)(\zeta) + v\,\mathrm{Im}\,\Phi(P)(\zeta)$, $(u, v) \neq (0, 0)$.

If $q \geqslant 1$, fix all roots of P except $(\varphi, \bar{\varphi}) \in Q^*$. Thus $L_{u,v}$ defines a linear form L_1 in $\varphi + \bar{\varphi}$ and $\varphi\bar{\varphi}$. For almost all choices of (u, v) one has $L_1 \not\equiv 0$. Indeed, otherwise $\Phi(P)(\zeta) = 0$ for all choices of φ; choose $|\varphi|$ close to 0 (by Lemma 1.5 and the two lines after its proof, $\Phi(P) \in \overline{V_{n-1}}$ – a contradiction) or tending to ∞ (hence $(\Phi(P))^R$ has a root close to 0 and in the same way $(\Phi(P))^R \in \overline{V_{n-1}}$, hence $\Phi(P) \in \overline{V_{n-1}}$).

The condition $L_{u,v} = 0$ implies that the root of $\Phi(P)$ close to ζ belongs to a smooth curve passing through ζ. Choose (u, v) such that at ζ this curve be nonparallel to the imaginary axis. The choice is possible due to $(\Phi(P))'(\zeta) \neq 0$. Vary φ so that $\mathrm{Re}\,\zeta$ decrease. Variation is possible until either (1) the couple $(\varphi, \bar{\varphi})$ becomes purely imaginary or a multiple positive root, or (2) $|\varphi|$ tends to 0 or to ∞. We saw above that (2) is impossible. If (1) takes place, then q decreases. If ζ becomes real, this is Case E.

If $q = h = 0$, then $\Phi(P) \in \overline{V_{n-1}}$ (part (2) of Remark 1). If $q = 0$, $g \geqslant 1$, $h \geqslant 1$, then set $P = (x - \eta)(x^2 + r)P_2$, $\eta > 0$, $r > 0$. The condition $L_1 = 0$ reads $h_1\eta r + h_2\eta + h_3 r + h_4 = 0$ $(*)$, where $(h_1, h_2, h_3, h_4) \neq (0, 0, 0, 0)$. If $h_1 = h_2 = 0$, then $r = -h_4/h_3 = \mathrm{const}$, L_1 is independent of η. Let η tend to 0 – this is the impossible situation (2). In the same way $h_3 = h_4 = 0$ (i.e. $\eta = -h_2/h_1$) is impossible. If $h_3/h_1 = h_4/h_2$, then $(*)$ becomes $(h_1 r + h_2)(h_1\eta + h_3) = 0$, i.e. $\eta = -h_2/h_1$ or $r = -h_4/h_3$ – impossible. If not write $\eta = (-h_3 r - h_4)/(h_1 r + h_2)$. Vary r towards an extremity of an interval, where η is defined, i.e. towards ∞, 0 or $-h_2/h_1$. In the last case $\eta \to \infty$. In all cases one is in the impossible situation (2).

Case C: *$\Phi(P)$ has a simple negative root η.* Reasoning analogous and simpler than in Case B.

Case D: *$\Phi(P)$ has a complex couple $(\zeta, \bar{\zeta})$ of multiplicity $m > 1$.* Consider ζ as a simple root of $\Phi(P)^{(m-1)}$. As in Case B decrease $\mathrm{Re}\,\zeta$ by varying Q. There is a root of $\Phi(P)$ close to ζ whose real part decreases. If ζ does not split, this is ζ; if it does, then the roots emerging from it are the ones of a perturbation of $(x - \zeta)^m$. For small values of the perturbation parameter one of them has real part $< \mathrm{Re}\,\zeta$.

Case E: *$\Phi(P)$ has an m-fold negative root.* Reasoning analogous and simpler than in Case D.

4^0. Suppose that $P = (x + 1)xS_1$, $(x + 1)S_1 \in U_{n-2}$. Use Lemma 1.5. Set $v_i = ((n - 1)a_i + 1)/n$, $\beta_i = (n - 1)a_i/n$ (hence $v_i = \beta_i + 1/n$), $\delta_j(v) := \sum_{2 \leqslant i_1 < \cdots < i_j \leqslant n-1} v_{i_1} \cdots v_{i_j}$. Notice that as $a_1 = 0$, one has $\delta_j(a) = \sigma_j$. Recall that $(c_1, \ldots, c_{n-2}) \in U_{n-2}$. By inductive assumption $(\delta_1(v), \ldots, \delta_{n-2}(v)) \in U_{n-2}$. For $\nu = 1, \ldots, n - 2$ there exist constants $r_{\nu,i} > 0$ such that $\delta_\nu(\beta) = \delta_\nu(v) + \sum_{l=1}^{\nu-1} (-1)^l r_{\nu,\nu-l}\delta_{\nu-l}(v)$. Hence $\mathrm{sign}(\delta_\nu(\beta)) = \mathrm{sign}(\delta_\nu(a)) = \mathrm{sign}(\delta_\nu(v))$, $(\delta_1(\beta), \ldots, \delta_{n-2}(\beta)) \in U_{n-2}$ and $(\delta_1(a), \ldots, \delta_{n-2}(a)) \in U_{n-2}$, i.e. $\Phi(P) \in (U_{n-1} \cap \{c_{n-1} = 0\}) \subset \partial U_{n-1}$.

Lemma 1.6. *The image under Φ of the half-line in $\mathcal{R}$ defined by the family of polynomials $M_t := (x + 1)(x^{n-1} + (-1)^{n-1}t)$, $t \geqslant 0$, is a half-line $l_{n-1} \subset \overline{U_{n-1}}$ beginning at $\overline{U_{n-1}} \cap \{c_{n-1} = 0\}$ its other points being interior for U_{n-1}.*

5^0. The lemma implies what remains to be proved of parts (2) and (3). Indeed, ∂U_{n-1} consists of the closures of $n - 1$ faces of dimension $n - 2$ spanned each by $n - 2$ coordinate half-axes (positive or negative is defined by the coordinate's index). The images of $n - 2$ of the $n - 1$ coordinate half-axes are half-lines $l_1, \ldots, l_{n-2}$ belonging to the interior of $\overline{U_{n-1}} \cap \{c_{n-1} = 0\}$ (follows from 4^0). They span $\Phi(\overline{U_{n-1}} \cap \{c_{n-1} = 0\})$. The other faces forming $\partial(\Phi(\overline{U_{n-1}}))$ are spanned by l_{n-1} and by $n - 3$ of the half-lines $l_1, \ldots, l_{n-2}$. By Lemma 1.6 these faces belong to $\overline{U_{n-1}}$, their intersections with $\{c_{n-1} = 0\}$ are the $(n - 3)$-dimensional faces of $\partial(\Phi(\overline{U_{n-1}} \cap \{c_{n-1} = 0\}))$. Any half-line in $\overline{U_{n-1}}$ beginning at $\{c_{n-1} = 0\}$ and parallel to Oc_{n-1} is mapped by Φ onto a half-line parallel to l_{n-1} (Φ is affine), beginning at $\{c_{n-1} = 0\}$ and belonging to $\{(-1)^{n-1}c_{n-1} \geqslant 0\}$ (by continuity). Thus $\Phi(U_{n-1}) \subset U_{n-1}$ which proves part (3).

Proof of Lemma 1.6. As Φ is affine, l_{n-1} is a half-line (beginning at $\{c_{n-1} = 0\}$ because if $t = 0$, then $a_1 = 0$, hence $\sigma_{n-1} = 0$; see Remark 1). As $M_t = x^n + x^{n-1} + (-1)^{n-1}tx + (-1)^{n-1}t$, one has $a_j = b_{j-1}$, $j = 3, \ldots, n - 1$ (Remark 1). When $t \to \infty$, the polynomial $(-1)^{n-1}M_t/t$ is a perturbation of $x + 1$ for which $a_j = b_{j-1}$, $j = 3, \ldots, n - 1$, $a_1 = b_n = \infty$, $a_2 = b_{n-1} = -(n - 1)$ (Remark 1). Hence for $t > 0$ large enough, a_1 and a_2 are close respectively to ∞ and b_{n-1}. The numbers a_i are roots of a real polynomial, so they are all real. As $a_1a_2 = t > 0$, one has $a_1 < 0$. Hence for $t > 0$ the vector $\tilde{\sigma}$ defines a point in U_{n-1}. □

6^0. Set $P := \sum_{j=0}^{n-1} \theta_j G_j$, see Remark 2. Denote by ν the greatest j for which $\theta_j \neq 0$. If $\nu = 0$ or 1, then $\Phi(P) = P = (x+1)^{n-1}(\theta_1 x + \theta_1 + \theta_0) \in \Pi_{n-1}$. If $\nu > 1$, as $1 = \lambda_0 = \lambda_1 < \cdots < \lambda_\nu$, one has $\Phi^k(P)/(x+1) = (\lambda_\nu)^k(\theta_\nu G_\nu + \sum_{j=0}^{\nu-1} \tilde{\theta}_{j,k} G_j)$, where $\lim_{k\to\infty} \tilde{\theta}_{j,k} = 0$, i.e. $J := \Phi^k(P)/((\lambda_\nu)^k (x+1)^{n-\nu})$ is a perturbation of $\theta_\nu G_\nu/(x+1)^{n-\nu}$ whose roots are real and distinct, so for k large enough $J \in \Pi_{n-1}$.

7^0. Suppose that $\overline{U_{n-1}} \ni S \neq x^{n-1}$. Present the $(n-1)$-tuple $\tilde{c}$ in the form $\rho\tilde{c}'$, where $\rho > 0$ and $\tilde{c}' \in \mathcal{S} \cap U_{n-1}$ ($\mathcal{S}$ is the unit sphere in $\mathbf{R}^{n-1}$). Recall that $(-1)^j c'_j \geqslant 0$, $j = 1, \ldots, n-1$.

Lemma 1.7. *There exists $\mu' \in \mathbf{N}$ depending only on n such that for each polynomial $P^* := (x+1)(c'_1 x^{n-2} + \cdots + c'_{n-1})$ (with $\tilde{c}'$ as above), $P^* \not\equiv 0$, one has $\Phi^{\mu'}(P^*) \in \Pi_{n-1}$.*

8^0. Deduce part (5) from the lemma. Enlarge μ' to assume that all roots of $\Phi^{\mu'}(P^*)$ are $\geqslant 0$ and distinct (indeed, $\Phi^{\mu'}(P^*) \in \overline{U_{n-2}}$ (part (2) of the theorem); $\Phi^{\mu'}(P^*)$ has no roots < 0 (Descartes rule); then use (2) of Remark 1). So for each $\tilde{c}' \in (\mathcal{S} \cap \overline{U_{n-1}})$ one can find an ε-neighbourhood $\tilde{\omega} \subset \mathcal{S}$ and $\rho > 0$ so large that $\Phi^{\mu'}(T) \in \Pi_{n-1}$ for all $T = \tilde{P} + x^{n-1}(x+1)/\tilde{\rho}$, $\tilde{P} \in \tilde{\omega}$, $\tilde{\rho} \geqslant \rho$. These ε-neighbourhoods cover $\mathcal{S} \cap \overline{U_{n-1}}$. For a finite subcovering denote by ρ_1 the maximal of the numbers ρ. Hence $\Phi^{\mu'}(P) \in \Pi_{n-1}$ for $P \in \overline{U_{n-1}}$ with $\rho \geqslant \rho_1$. The set $\Lambda := \{P \in \overline{U_{n-1}} \mid \rho \leqslant \rho_1\}$ is compact. For each $P \in \Lambda$ find $\mu_1 \in \mathbf{N}$ and an ε_1-neighbourhood ω such that $\Phi^{\mu_1}(P) \in \Pi_{n-1}$ for all $P \in \Lambda \cap \omega$. Fix a finite ε_1-subcovering of Λ, denote by μ'' the biggest of its numbers μ_1. Set $\mu = \max(\mu', \mu'')$. Part (5) of the theorem is proved. □

Proof of Lemma 1.7. Set $\Sigma_r := \{\tilde{c}' \mid c'_1 = \cdots = c'_r = 0\}$, $0 \leqslant r \leqslant n-2$, $\Theta_r := \Sigma_r \setminus \Sigma_{r+1}$. Proof by induction on $n-1-r$. Induction base: if $\tilde{c}' \in \Theta_{n-2}$, then $P = (x+1)c'_{n-2}$ and $\Phi(P) \in \Pi_{n-1}$, see part (1) of Remark 1.

Suppose that there exists $\mu^* \in \mathbf{N}$ such that for every polynomial $P \in \Sigma_r$ (i.e. $\tilde{c}' \in \Sigma_r$) one has $\Phi^{\mu^*}(P) \in \Pi_{n-1}$. Enlarge μ^* to assume that all roots of $\Phi^{\mu^*}(P)$ are nonnegative and distinct (Remark 1).

If $P \in \Theta_{r-1}$, consider $P/c'_r = (x+1)(x^r + d_1 x^{r-1} + \cdots + d_r)$, $d_i = c'_{r+i}/c'_r$. Set $(d_1, \ldots, d_r) =: \tilde{d} := \rho\tilde{d}'$, $\rho > 0$, $(d'_1, \ldots, d'_r) =: \tilde{d}' \in \mathcal{S}^0 \cap \overline{U_r}$ ($\mathcal{S}^0$ is the unit sphere in $\mathbf{R}^r$). For each $\tilde{d}'$ find $\varepsilon_2 > 0$ so small and $\rho_2 > 0$ so large that $\Phi^{\mu^*}(P) \in \Pi_{n-1}$ in an ε_2-neighbourhood of $\tilde{d}' \in \mathcal{S}^0 \cap \overline{U_r}$ and for $\rho \geqslant \rho_2$. Choose a finite subcovering of $\mathcal{S}^0 \cap \overline{U_r}$ by ε_2-neighbourhoods, denote by ρ_2^* the greatest of the numbers ρ_2.

The set $\mathcal{Y} = \{\tilde{d} \mid P \in \Theta_{r-1}, \rho \leqslant \rho_2^*\}$ is compact. For each point of it find $\mu_3^* \in \mathbf{N}$ and an ε_3-neighbourhood Ω such that $\Phi^{\mu_3^*}(P) \in \Pi_{n-1}$ for $P \in \Omega \cap \mathcal{Y}$. Find a finite ε_3-subcovering of $\mathcal{Y}$, denote by μ_4^* the largest of the numbers μ_3^*. Set $\tilde{\mu} := \max(\mu^*, \mu_4^*)$. Hence $\Phi^{\tilde{\mu}}(P) \in \Pi_{n-1}$ for $P \in \Theta_{r-1} \cup \Sigma_r = \Sigma_{r-1}$. □

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