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C. R. Acad. Sci. Paris, Ser. I 347 (2009) 1403–1407
Comptes Rendus Mathematique

Algebraic Geometry/Topology

Rigidity for equivariant K-theory

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Received 10 October 2008; accepted after revision 27 October 2009

Available online 7 November 2009

Presented by Christophe Soulé

Abstract

We extend the classical rigidity results for K-theory to the equivariant setting of linear algebraic group actions. These results concern rigidity for rational points, field extensions, and Hensel local rings. ***To cite this article: S. Yagunov, P.A. Østvær, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***
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Résumé

Théorèmes de rigidité classiques pour la K-théorie. Nous étendons les théorèmes de rigidité classiques pour la K-théorie au cadre équivariant de actions des groupes algébriques linéaire. Ces résultats concernent la rigidité pour les points rationels, les extensions de corps et les anneaux locaux henséliens. ***Pour citer cet article : S. Yagunov, P.A. Østvær, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***
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Version française abrégée

Soit $\mathcal{F}$ un foncteur contravariant de la catégorie des schémas lisses quasi-projectifs de type fini sur un corps infini k à valeurs dans la catégorie des modules. La *propriété de rigidité* est vérifiée pour $\mathcal{F}$ si, pour tout schéma intègre X de type spécifié, deux sections $\sigma_0, \sigma_1 : \operatorname{Spec} k \to X$ du morphisme structural $X \to \operatorname{Spec} k$ induisent des homomorphismes égaux $\sigma_0^* = \sigma_1^* : \mathcal{F}(X) \to \mathcal{F}(\operatorname{Spec} k)$.

Pour un groupe algébrique linéaire G sur k et un k-schéma G-équivariant de type fini V, prenons pour $\mathcal{F}$ le K-foncteur G-équivariant à coefficients finis $\mathsf{K}_*(\mathsf{G}, V \times_k -; \mathbf{Z}/n)$, où $(n, \operatorname{Char} k) = 1$. Nous montrons que la propriété de rigidité est vérifiée pour un tel foncteur $\mathcal{F}$ si et seulement si elle est vérifiée dans le cas particulier où X est une droite affine munie des sections induites par les points 0 et 1.

Une conséquence de ce théorème est que si V est lisse, alors notre foncteur est invariant pour les changements de base suivants : extensions de corps algébriquement clos et déformations infinitésimales (i.e. morphismes des anneaux locaux henséliens vers leur corps résiduels).

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doi:10.1016/j.crma.2009.10.020

1. Introduction

For k an infinite field and G a linear algebraic group over k, let Sm_k denote the category of smooth quasi-projective k-schemes of finite type and fix some V in the category $\mathsf{Sch}_k^{\mathsf{G}}$ of G-equivariant separated k-schemes of finite type and G-maps. The latter data entails a G-action $\sigma : V \times \mathsf{G} \to V$ subject to the usual associative and unital identities for a group action. The abelian subcategory of G-modules on V comprising coherent $\mathcal{O}_V$-modules and its exact subcategory of locally free $\mathcal{O}_V$-modules, a.k.a. G-vector bundles on V, gives rise to the G-equivariant K'-theory (respectively K-theory) of V. When G is the trivial group, this reduces to ordinary K'- and K-theory. Throughout we consider mod-n coefficients for $n > 0$ relatively prime to the exponential characteristic of k, and assume that G acts trivially on the base fields.

Theorem *(Rigidity for rational points). Suppose the field k is algebraically closed. Then the following statements are equivalent*:

(1) *For every connected X in Sm_k with trivial G-action and any two rational points x_0 and x_1 on X,*

$$x_0^* = x_1^* : \mathsf{K}_*(\mathsf{G}, V \times_k X; \mathbf{Z}/n) \rightrightarrows \mathsf{K}_*(\mathsf{G}, V; \mathbf{Z}/n).$$

(2) *The rational points 0 and 1 on the affine line $\mathbf{A}_k^1$ with trivial G-action yield equal pullback maps*

$$\mathsf{K}_*\big(\mathsf{G}, V \times_k \mathbf{A}_k^1; \mathbf{Z}/n\big) \rightrightarrows \mathsf{K}_*(\mathsf{G}, V; \mathbf{Z}/n).$$

The four step proof of the theorem consists of checking that $\mathsf{K}_*(\mathsf{G}, V \times_k -; \mathbf{Z}/n)$ defines a functor with weak transfers on Sm_k in the sense of [9].

If V is smooth, so that the naturally induced map $\mathsf{K}_*(\mathsf{G}, V \times_k X) \to \mathsf{K}'_*(\mathsf{G}, V \times_k X)$ is an isomorphism for all X in Sm_k due to [14, Theorem 5.7 and Corollary 5.8], then the second condition in the theorem holds because K'-theory is homotopy invariant [14, Theorem 4.1], cf. Lemma 2.3. Although G acts trivially on X in the previous theorem we do not assume that G acts trivially on V. The following result generalizes the main theorem in [11]:

Theorem *(Rigidity for field extensions). Suppose K/k is an extension of separably closed fields, $(V, \sigma) \in \mathsf{Sch}_k^{\mathsf{G}}$ is smooth, and $n > 0$ is as above. With the induced G-action $\sigma \times \mathrm{id}_K$ on the base change scheme $V_K \equiv V \times_k K$ the natural G-map $V_K \to V$ induces an isomorphism*

$$\mathsf{K}_*(\mathsf{G}, V; \mathbf{Z}/n) \xrightarrow{\cong} \mathsf{K}_*(\mathsf{G}, V_K; \mathbf{Z}/n).$$

Proof. To begin with we assume that the fields are algebraically closed. If a Noetherian G-scheme X is the inverse limit of a system $\{X_\alpha\}$ of Noetherian G-schemes with flat affine transition maps, then $\mathsf{K}'_*(\mathsf{G}, X) \cong \mathrm{colim}_\alpha \mathsf{K}'_*(\mathsf{G}, X_\alpha)$, as Thomason noted in [15, §3.7] (in fact, the category of coherent G-modules on X is the 2-colimit of the categories of coherent G-modules on the X_α's). Our claim follows now as in the proof of [9, Theorem 1.14] by combining rigidity for rational points and Lemma 2.3 (viewing $\operatorname{Spec} K$ as an inverse limit of smooth affine k-schemes with trivial G-actions). The argument in [10, §7, Proposition 4.8] shows that purely inseparable extensions of degree p^d induce isomorphisms on $\mathsf{K}_*(\mathsf{G}, -; \mathbf{Z}/n)$ with $(n, p) = 1$. Replacing separably closed field with their algebraic closures completes the proof. □

The last of our main results deals with rigidity for equivariant K-theory of Hensel local rings. For non-equivariant results we refer to [2,3,13].

Theorem *(Rigidity for Hensel local rings). Suppose k is an infinite field, $(V, \sigma) \in \mathsf{Sch}_k^{\mathsf{G}}$ is smooth, and $n > 0$ is as above. If $P \in X(k)$ is a rational point on $X \in \mathsf{Sm}$, let $\mathcal{O}_{X,P}^h$ denote the corresponding Hensel local ring. Then there is a natural isomorphism*

$$\mathsf{K}_*\big(\mathsf{G}, V \times_k \mathcal{O}_{X,P}^h; \mathbf{Z}/n\big) \xrightarrow{\cong} \mathsf{K}_*(\mathsf{G}, V; \mathbf{Z}/n).$$

In order to prove this result we basically check that the conditions in [6] are satisfied.

If G is reductive and k has characteristic zero, then representations of G are direct sums of irreducible representations. It follows that there exists an isomorphism between $\mathsf{K}_*(G,k)$ and $\mathsf{K}_*(k)\otimes_{\mathbf{Z}} R(G)$, where $R(G)$ denotes the representation ring of G. Hence Suslin's computation of the K-groups $\mathsf{K}_*(k;\mathbf{Z}/n)$ [12] implies there is an isomorphism $\mathsf{K}_*(G,k;\mathbf{Z}/n)\cong R(G)/n[\beta]$. Here β is the Bott element in $\mathsf{K}_*(k;\mathbf{Z}/n)$ of degree 2. Using rigidity for field extensions, as stated in the above, Krishna has deduced a more general computation in the recent work [7].

2. Proofs

Proof *(Rigidity for rational points)***.** For legibility, let $\mathcal{P}^{\mathsf{G}}$ denote the category of G-vector bundles of finite rank and BQ the classifying space of Quillen's Q-construction [10]. We write

$$\mathcal{F}(-)\equiv\pi_{*+1}\big(\mathsf{BQ}\,\mathcal{P}^{\mathsf{G}}(V\times_k -);\mathbf{Z}/n\big):(\mathsf{Sm}_k)^{\mathrm{op}}\to\mathbf{Ab}$$

for the functor $\mathsf{K}_*(\mathsf{G},V\times_k -;\mathbf{Z}/n)$ in the formulation of the theorem. Assuming that (2) holds, we show $\mathcal{F}(-)$ acquires weak transfers for the class $C_{\mathtt{ff}}$ of finite flat maps in the sense of [9].

Lemma 2.1. *If $f:X\to Y$ is a finite flat map in Sm_k then the direct image functor induces a transfer map $f_*:\mathcal{F}(X)\to\mathcal{F}(Y)$.*

Proof. The assumption on f implies that $F=\mathrm{id}_V\times_k f:V\times_k X\to V\times_k Y$ is a finite flat map. This shows [4, Theorem 3.2.1, Corollary 1.3.2] the direct image map F_* is an exact functor on the category of coherent G-sheaves on X. In this situation it suffices to verify that F_* preserves locally free sheaves. The latter statement is local, so it suffices to consider affine schemes. Given a Noetherian ring R, a finite flat algebra S over R and a projective S-module P, we wish to show that P is a projective R-module. It suffices to show that $\mathrm{Hom}_R(P,-)\simeq\mathrm{Hom}_S(P,\mathrm{Hom}_R(S,-))$ is an exact functor. This follows by combining the standard facts that S is a projective R-module according to [8, I, Theorem 2.9], P is a projective S-module and the composition of two exact functors is an exact functor. □

Remark 1. Since we require transfer maps for a very restricted class of morphisms, we may avoid using higher direct images and Tor-formulas as in Quillen's general transfer construction [10, §7].

Secondly, we need to verify the additivity, base change, and normalization conditions formulated in [9].

Remark 2. For our purposes it suffices to check the base change property only for closed embeddings. Therefore, our form of this property is less general than in [9].

(i) Additivity: For $X=X_0\sqcup X_1$ with corresponding embeddings $i_m:X_m\hookrightarrow X$ for $m=0,1$ and $f:X\to Y$ a map in Sm_k, then

$$f_*=(fi_0)_*i_0^*+(fi_1)_*i_1^*.$$

(ii) Base change: For every Cartesian square

$$\begin{tikzcd} X' \arrow[r, hook, "\tilde{g}"] \arrow[d, "\tilde{f}"'] & Y' \arrow[d, "f"] \\ X \arrow[r, hook, "g"] & Y \end{tikzcd}$$

where $f\in C_{\mathtt{ff}}$ and g is a closed embedding, one has $g^*f_*=\tilde{f}_*\tilde{g}^*$.

(iii) Normalization: If f is the identity map on k, then $f_*=\mathrm{id}_{\mathcal{F}(k)}$.

The direct and inverse image functors are natural G-maps so that the isomorphisms appearing in the proof of the base change property below are in fact G-isomorphisms. Thus the additivity condition follows immediately by using additivity of the direct image functor.

Next we show that $g^* f_* = \tilde{f}_* \tilde{g}^*$ for the maps in the Cartesian diagram. To wit, for a locally free sheaf $\mathcal{S}$ on Y', the adjunction between left adjoint inverse image functors and right adjoint direct image functors furnishes canonical elements $\alpha \in \mathrm{Hom}_{Y'}(\mathcal{S}, \tilde{g}_*\tilde{g}^*(\mathcal{S}))$ and $\beta \in \mathrm{Hom}_{Y'}(f^* f_*(\mathcal{S}), \mathcal{S})$. Moreover, there are canonical isomorphisms

$$\begin{aligned}\mathrm{Hom}_{Y'}\big(f^* f_*(\mathcal{S}), \tilde{g}_*\tilde{g}^*(\mathcal{S})\big) &\cong \mathrm{Hom}_{X'}\big(\tilde{g}^* f^* f_*(\mathcal{S}), \tilde{g}^*(\mathcal{S})\big)\\ &\cong \mathrm{Hom}_{X'}\big(\tilde{f}^* g^* f_*(\mathcal{S}), \tilde{g}^*(\mathcal{S})\big)\\ &\cong \mathrm{Hom}_{X}\big(g^* f_*(\mathcal{S}), \tilde{f}_*\tilde{g}^*(\mathcal{S})\big).\end{aligned}$$

The image of the composite $\alpha\beta \in \mathrm{Hom}_{Y'}(f^* f_*(\mathcal{S}), \tilde{g}_*\tilde{g}^*(\mathcal{S}))$ under these isomorphisms determines a map of sheaves on X

$$\widetilde{\alpha\beta} : g^* f_*(\mathcal{S}) \longrightarrow \tilde{f}_*\tilde{g}^*(\mathcal{S}).$$

Using the assumptions on f and g one verifies easily that $\widetilde{\alpha\beta}$ is a fiberwise isomorphism. Hence $\widetilde{\alpha\beta}$ is a sheaf isomorphism. This completes the proof of the base change property.

Finally, the normalization condition holds trivially.

In order to finish the proof it remains to note that the maps appearing in [9, §1] belong to $C_{\mathtt{ff}}$, and moreover that the base change diagrams in [9, §1] are of the type above (with respect to some closed embedding). These conditions hold according to the following elementary result:

Lemma 2.2. *If X is a smooth projective irreducible curve over a field and $f : X \to \mathbf{P}^1$ a dominant morphism (i.e. a non-constant rational function on X), then $f \in C_{\mathtt{ff}}$.*

Proof. This is a special case of [5, III, Proposition 9.7]. □

The proof of rigidity for rational points is now complete. □

The second condition in the rigidity for rational points result serves as a "replacement" for homotopy invariance. A functor $F : (\mathsf{Sm}_k^{\mathsf{G}})^{\mathrm{op}} \to \mathbf{Ab}$ is homotopy invariant for X if the canonical projection map $p : X \times_k \mathbf{A}^1_k \to X$ induces an isomorphism $p^* : F(X) \to F(X \times_k \mathbf{A}^1_k)$. (Here and below G acts trivially on the affine line $\mathbf{A}^1_k$.)

Lemma 2.3.

(1) *If F is homotopy invariant for X, then the rational points* 0 *and* 1 *on the affine line over k yield equal pullback maps*

$$F(X \times_k \mathbf{A}^1_k) \rightrightarrows F(X).$$

(2) *If*

$$i_0^* = i_1^* : F(Y \times_k \mathbf{A}^1_k) \rightrightarrows F(Y)$$

holds for $Y = X \times_k \mathbf{A}^1_k$, then F is homotopy invariant for X.

Proof. Part (1) holds since the composite map $X \xrightarrow{i} Y \xrightarrow{p} X$ is the identity. In order to prove (2) we use the diagram

$$F\big(X \times_k \mathbf{A}^1_k\big) \xrightarrow{\mu^*} F\big(X \times_k \mathbf{A}^1_k \times_k \mathbf{A}^1_k\big) \overset{i_0^*}{\underset{i_1^*}{\rightrightarrows}} F\big(X \times_k \mathbf{A}^1_k\big).$$

Here μ^* is induced by the product map $\mu : \mathbf{A}^1_k \times_k \mathbf{A}^1_k \to \mathbf{A}^1_k$ while the maps i_0^* and i_1^* are induced by the rational points 0 and 1 on $\mathbf{A}^1_k$. By hypothesis, the composite maps in the diagram coincide. The map involving i_1^* is the identity and $i_0\mu$ equals the composite $Y \xrightarrow{p} X \xrightarrow{i} Y$. Hence i^* is inverse to p^*. □

As noted in the introduction, the proof of rigidity for extensions is now complete.

Proof *(Rigidity for Hensel local rings)***.** The Additivity and Normalization properties stated in [6] coincide with the those this Note, while the base change diagrams in [6] are also of the type considered here (with respect to some closed embedding), cf. Remark 2. This shows that with our formulation of the base change property, the approach in [6] can be adopted verbatim. In [6] one also requires transfer maps (trace homomorphisms) for finite separable field extensions. It remains to note that such extensions induce maps between schemes in $C_{\mathtt{ff}}$. Besides these conditions one should also verify validity of Gersten's conjecture in the equivariant case. This can be done following the approach of [1]. □

Acknowledgements

This work was mostly done during a stay of the first author at the Max-Planck-Institut für Mathematik (Bonn). He is deeply grateful to the institute for its hospitality and perfect working conditions. He also thanks the University of Oslo for financial support during his short visit to Norway.

Both authors are grateful to Max Karoubi, Burt Totaro, and the referee for valuable comments on the Note. We also wish to thank Tatiana Smirnova-Nagnibeda for her help with translating part of this paper into French.

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