Available online at www.sciencedirect.com
ScienceDirect
C. R. Acad. Sci. Paris, Ser. I 347 (2009) 1415–1418
COMPTES RENDUS MATHEMATIQUE

Topology

The image of Singer's fourth transfer ☆

Nguyễn H.V. Hưng, Võ T.N. Quỳnh

Department of Mathematics, Vietnam National University, Hanoi, 334 Nguyễn Trãi Street, Hanoi, Vietnam

Received 13 April 2009; accepted after revision 14 October 2009

Available online 12 November 2009

Presented by Christophe Soulé

Dedicated to Haynes R. Miller on the occasion of his sixtieth birthday

Abstract

We complete in this Note the description of Singer's fourth transfer, already studied by many authors. More precisely, we show that each element of the family $\{p_i \mid i \geqslant 0\}$ belongs to the image of this fourth transfer. Combining this with previous results by R. Bruner, L.M. Hà, T.N. Nam and the first author, we deduce that the image of the algebraic transfer contains all the elements of the families $\{d_i \mid i \geqslant 0\}$, $\{e_i \mid i \geqslant 0\}$, $\{f_i \mid i \geqslant 0\}$ and $\{p_i \mid i \geqslant 0\}$, but none from the families $\{g_i \mid i \geqslant 1\}$, $\{D_3(i) \mid i \geqslant 0\}$ and $\{p'_i \mid i \geqslant 0\}$.

The method used to prove that elements are in the transfer's image can be applied not only to the family of p_i's but to the families of d_i's, e_i's and f_i's as well. ***To cite this article: N.H.V. Hưng, V.T.N. Quỳnh, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

L'image du quatrième transfert de Singer. Dans cette Note on achève la description du quatriéme transfert de Singer, complétant ainsi le travail de nombreux auteurs. Plus précisement on montre que chaque élément de la famille $\{p_i \mid i \geqslant 0\}$ appartient à l'image du quatriéme transfert. Combinant cela avec des résultats antérieurs de R. Bruner, L.M. Hà, T.N. Nam, et du premier auteur, on en déduit que l'image du transfert algébrique contient chaque élément des quatre familles $\{d_i \mid i \geqslant 0\}$, $\{e_i \mid i \geqslant 0\}$, $\{f_i \mid i \geqslant 0\}$, et $\{p_i \mid i \geqslant 0\}$, et ne contient aucun élément des trois familles $\{g_i \mid i \geqslant 1\}$, $\{D_3(i) \mid i \geqslant 0\}$, and $\{p'_i \mid i \geqslant 0\}$.

La méthode utilisée pour montrer que des éléments sont dans l'image du transfert peut être appliquée non seulement à la famille p_i mais aussi aux familles d_i, e_i, and f_i. ***Pour citer cet article : N.H.V. Hưng, V.T.N. Quỳnh, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

1. Statement of results

Let $H_*(X)$ denote the mod 2 homology of a space X. Let now $\mathbb{V}_s$ be an s-dimensional $\mathbb{F}_2$-vector space, and $PH_*(B\mathbb{V}_s)$ the primitive subspace consisting of all elements in $H_*(B\mathbb{V}_s)$, which are annihilated by every positive-degree operation in the mod 2 Steenrod algebra, $\mathcal{A}$. The general linear group $GL_s := GL(\mathbb{V}_s)$ acts regularly on the

☆ The work was supported in part by a grant of the NAFOSTED.
E-mail addresses: nhvhung@vnu.edu.vn (N.H.V. Hưng), quynhvtn@vnu.edu.vn (V.T.N. Quỳnh).

1631-073X/$ – see front matter
doi:10.1016/j.crma.2009.10.018

classifying space $B\mathbb{V}_s$ and thus on the homology $H_*(B\mathbb{V}_s)$. Since the two actions of $\mathcal{A}$ and GL_s upon $H_*(B\mathbb{V}_s)$ commute with each other, there is an inherited action of GL_s on $PH_*(B\mathbb{V}_s)$. In [14], W. Singer defined a homomorphism

$$\widetilde{\mathrm{Tr}}_s : PH_d(B\mathbb{V}_s) \to \mathrm{Ext}_{\mathcal{A}}^{s,s+d}(\mathbb{F}_2, \mathbb{F}_2),$$

and showed that this map factors through the quotient of its domain's GL_s-coinvariants to give rise the so-called algebraic transfer

$$\mathrm{Tr}_s : \mathbb{F}_2 \underset{GL_s}{\otimes} PH_d(B\mathbb{V}_s) \to \mathrm{Ext}_{\mathcal{A}}^{s,s+d}(\mathbb{F}_2, \mathbb{F}_2).$$

This is an algebraic version of the geometrical transfer $\mathrm{tr}_s : \pi_*^S((B\mathbb{V}_s)_+) \to \pi_*^S(S^0)$ to the stable homotopy groups of spheres ([6]).

It has been proved that Tr_s is an isomorphism for $s = 1, 2$ by Singer [14] and for $s = 3$ by Boardman [1]. Among other things, these data together with the fact that $\mathrm{Tr} = \bigoplus_s \mathrm{Tr}_s$ is an algebra homomorphism (see [14]) show that Tr_s is highly nontrivial. Therefore, the algebraic transfer is expected to be a useful tool in the study of the mysterious cohomology of the Steenrod algebra, $\mathrm{Ext}_{\mathcal{A}}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$.

According to W.H. Lin and M. Mahowald [8], $\mathrm{Ext}_{\mathcal{A}}^{4,*}(\mathbb{F}_2, \mathbb{F}_2)$ contains seven Sq^0-families of indecomposable elements, namely d_i, e_i, f_i, g_i, p_i, $D_3(i)$, and p'_i.

The following theorem states the main result of this Note:

Theorem 1.1. *Every element in the usual family* $\{p_i \mid i \geqslant 0\}$, *where*

$$p_i \in \mathrm{Ext}_{\mathcal{A}}^{4,2^{i+5}+2^{i+2}+2^i}(\mathbb{F}_2, \mathbb{F}_2), \quad i \geqslant 0,$$

belongs to the image of the fourth algebraic transfer, Tr_4.

It has been known that all the decomposable elements in the fourth cohomology group $\mathrm{Ext}_{\mathcal{A}}^{4,*}(\mathbb{F}_2, \mathbb{F}_2)$ belong to the image of the fourth algebraic transfer.

Combining the above theorem with some earlier results by R. Bruner, L.M. Hà, T.N. Nam, and the first named author, we obtain the following consequence that determines explicitly the image of the fourth algebraic transfer. It establishes a conjecture by the first named author in [5].

Corollary 1.2. *The image of the fourth algebraic transfer,* Tr_4, *contains every element in the four families* $\{d_i \mid i \geqslant 0\}$, $\{e_i \mid i \geqslant 0\}$, $\{f_i \mid i \geqslant 0\}$, *and* $\{p_i \mid i \geqslant 0\}$, *whereas it does not contain any element in the three families* $\{g_i \mid i \geqslant 1\}$, $\{D_3(i) \mid i \geqslant 0\}$, *and* $\{p'_i \mid i \geqslant 0\}$.

The result on $\{g_i \mid i \geqslant 1\}$ is due to R. Bruner, L.M. Hà, and the first named author [2]; that on $\{D_3(i) \mid i \geqslant 0\}$, and $\{p'_i \mid i \geqslant 0\}$ is due to the first named author [5]; the conclusion on $\{d_i \mid i \geqslant 0\}$, $\{e_i \mid i \geqslant 0\}$ is proved by L.M. Hà [3]; while that on $\{f_i \mid i \geqslant 0\}$ is showed by T.N. Nam [12].

It should be noted that the result by R. Bruner, L.M. Hà, and the first named author on the family $\{g_i \mid i \geqslant 1\}$, and the one by the first named author on the two families $\{D_3(i) \mid i \geqslant 0\}$, $\{p'_i \mid i \geqslant 0\}$ gave a negative answer to a conjecture of Minami [11] predicting that the localization of Tr_s given by inverting the squaring operation Sq^0 is an isomorphism.

W. Singer conjectured in [14] that the algebraic transfer is a monomorphism. We are confident that this prediction could be proved for the fourth transfer by using the result of the amazing 240-page paper by N. Sum [15] on the hit problem for the polynomial algebra of four variables.

To prove the main result, we find an explicit element $\tilde{p}_0 \in PH_*(B\mathbb{V}_4)$ such that

$$\widetilde{\mathrm{Tr}}_4(\widetilde{p}_0) = p_0.$$

Let $\overline{p}_0$ denote the image of the element $\widetilde{p}_0$ under the projection $\mathrm{pr} : PH_*(B\mathbb{V}_s) \to \mathbb{F}_2 \otimes_{GL_s} PH_*(B\mathbb{V}_s)$. We then have

$$\mathrm{Tr}_4(\overline{p}_0) = p_0.$$

Therefore, the main theorem is proved by the two facts that (1) through the algebraic transfer, the classical squaring operation Sq^0 on its target and the Kameko squaring operation Sq^0 on its domain commute with each other (see [1,11]), and that (2) the family $\{p_i \mid i \geqslant 0\}$ is an Sq^0-family initiated by p_0 (see [8]).

In order to make the Note self-contained, let us give definitions of the classical squaring operation and the Kameko squaring one.

Let $\mathcal{A}_*$ be the dual of the Steenrod algebra. The classical squaring operation $Sq^0 : \mathrm{Ext}_{\mathcal{A}}^{s,t}(\mathbb{F}_2, \mathbb{F}_2) \to \mathrm{Ext}_{\mathcal{A}}^{s,2t}(\mathbb{F}_2, \mathbb{F}_2)$ is the homomorphism induced in cohomology by the Frobenius map $F : \mathcal{A}_* \to \mathcal{A}_*$, $F(\xi) = \xi^2$. (See [9,10].)

Let $(x_1, \ldots, x_s)$ be a basis of the $\mathbb{F}_2$-vector space $H^1(B\mathbb{V}_s) \cong \mathrm{Hom}(\mathbb{V}_s, \mathbb{F}_2)$. In [7], Kameko defined a homomorphism

$$\widetilde{Sq^0} : H_*(B\mathbb{V}_s) \to H_*(B\mathbb{V}_s), \qquad a_1^{(i_1)} \cdots a_s^{(i_s)} \mapsto a_1^{(2i_1+1)} \cdots a_s^{(2i_s+1)},$$

where $a_1^{(i_1)} \cdots a_s^{(i_s)}$ is dual to $x_1^{i_1} \cdots x_s^{i_s}$ with respect to the basis of $H^*(B\mathbb{V}_s)$ consisting of all monomials in $x_1, \ldots, x_s$. He proved that this is a GL_s-homomorphism and maps $PH_*(B\mathbb{V}_s)$ to itself. The induced homomorphism $Sq^0 : \mathbb{F}_2 \otimes_{GL_s} PH_*(B\mathbb{V}_s) \to \mathbb{F}_2 \otimes_{GL_s} PH_*(B\mathbb{V}_s)$ is called the Kameko squaring operation.

Our method for showing some elements to be in the image of the transfer could be applied not only to the family p_i, but also to the families d_i, e_i, and f_i as well.

In [4], the first named author gave an explicit chain level representation for the dual Tr_s^* of the algebraic transfer, which maps from the s-grading submodule of the dual of the lambda algebra to $\mathbb{F}_2[x_1^{\pm 1}, \ldots, x_s^{\pm 1}]$, and evidently sends the submodule of cycles to $\mathbb{F}_2[x_1, \ldots, x_s]$. It should be interesting to apply this chain level representation in order to explicitly find the polynomials, which represent the images under Tr_4^* of the classes in $\mathrm{Tor}_4^{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$. This is an another way to determine the image of the algebraic transfer. We will return back to this problem in the near future.

2. Remarks

Our method for proving that $p_i \in \mathrm{Im}(\mathrm{Tr}_4)$ is rather similar to that by L.M. Hà in [3], where he showed that $d_0, e_0 \in \mathrm{Im}(\mathrm{Tr}_4)$ and $g_1 \notin \mathrm{Im}(\mathrm{Tr}_4)$. Indeed, he and we basically used the chain level representation for the algebraic transfer given by Boardman [1]. However, Hà additionally exploited Zachariou's and Palmieri's results on the restriction from the cohomology of the Steenrod algebra to the cohomology of its commutative sub-Hopf algebras.

Let $\xi_i \in \mathcal{A}_*$ be the degree $2^i - 1$ Milnor element, which is dual to $Sq^{2^{i-1}} \cdots Sq^2 Sq^1$ with respect to the admissible basis of the Steenrod algebra $\mathcal{A}$. Let h_{ij} be represented by $[\xi_i^{2^j}]$ in the cobar complex for $\mathcal{A}$. By Tangora [16], the elements $d_0, e_0, g_1 \in \mathrm{Ext}_{\mathcal{A}}^4(\mathbb{F}_2, \mathbb{F}_2)$ are represented by $b_{02}b_{12} + h_1^2 b_{03}$, $b_{12}h_0(1)$, and b_{12}^2 respectively. Note that Tangora's elements b_{ji}, h_i, $h_0(1)$ are denoted in this paper by h_{ij}^2, h_{1i}, $h_{11}h_{30} + h_{20}h_{21}$ respectively.

Let $E(2)$ be the commutative sub-Hopf algebra of $\mathcal{A}$ defined by

$$E(2)^* \cong \mathcal{A}_*/\big(\xi_1, \xi_2^4, \xi_3^4, \ldots\big),$$

whose cohomology is $\mathrm{Ext}_{E(2)}^*(\mathbb{F}_2, \mathbb{F}_2) \cong \mathbb{F}_2[h_{ij} \mid j < 2 \leqslant i] = \mathbb{F}_2[h_{20}, h_{21}, h_{30}, h_{31}, \ldots]$.

2.1. According to Zachariou [17], d_0, e_0 and g_1 have nonzero images under the restriction from the cohomology of the Steenrod algebra to that of $E(2)$. Indeed,

$$\mathrm{Res}(d_0) = \mathrm{Res}\big(b_{02}b_{12} + h_1^2 b_{03}\big) = \mathrm{Res}\big(h_{20}^2 h_{21}^2 + h_{11}^2 h_{30}^2\big) = h_{20}^2 h_{21}^2,$$
$$\mathrm{Res}(e_0) = \mathrm{Res}\big(b_{12}h_0(1)\big) = \mathrm{Res}\big(h_{21}^2(h_{11}h_{30} + h_{20}h_{21})\big) = h_{20}h_{21}^3,$$
$$\mathrm{Res}(g_1) = \mathrm{Res}\big(b_{12}^2\big) = \mathrm{Res}\big(h_{21}^4\big) = h_{21}^4,$$

as $\mathrm{Res}(h_{ij}) = 0$ for $i \leqslant j$ (see [13]).

2.2. By Palmieri [13], d_0, e_0 and g_1 are the only indecomposable elements in $\mathrm{Ext}_{\mathcal{A}}^4(\mathbb{F}_2, \mathbb{F}_2)$ whose images under the restriction are nonzero. Indeed, that the restriction vanishes on the 4 families f_i, p_i, $D_3(i)$, p_i' can directly be seen by combining the chain level representatives $f_0 = h_{12}^2 h_{30}^2$, $p_0 = h_{10}h_{13}h_{31}^2$, $D_3(0) = h_{14}h_0(1,2)$, $p_0' = h_{10}h_{14}h_{32}^2$, given in [16] and the fact that $\mathrm{Res}(h_{ij}) = 0$ for $i \leqslant j$. Following [10], the squaring operation is defined as follows

$$Sq^0\big([a_1|\cdots|a_s]\big) = \big[a_1^2|\cdots|a_s^2\big].$$

In particular, $Sq^0[\xi_i^{2^j}] = [\xi_i^{2^{j+1}}]$, or equivalently $Sq^0(h_{ij}) = h_{ij+1}$. Hence

$$\mathrm{Res}(d_1) = \mathrm{Res}\, Sq^0(d_0) = Sq^0\, \mathrm{Res}(d_0) = Sq^0\big(h_{20}^2 h_{21}^2\big) = h_{21}^2 h_{22}^2 = 0,$$
$$\mathrm{Res}(e_1) = \mathrm{Res}\, Sq^0(e_0) = Sq^0\, \mathrm{Res}(e_0) = Sq^0\big(h_{20} h_{21}^3\big) = h_{21} h_{22}^3 = 0,$$
$$\mathrm{Res}(g_2) = \mathrm{Res}\, Sq^0(g_1) = Sq^0\, \mathrm{Res}(g_1) = Sq^0\big(h_{21}^4\big) = h_{22}^4 = 0,$$

as $h_{22} = 0$ in the cohomology of $E(2)$ (see [13]). Since the restriction commutes with the squaring operation, we get

$$\mathrm{Res}(d_i) = 0, \qquad \mathrm{Res}(e_i) = 0, \qquad \mathrm{Res}(g_{i+1}) = 0,$$

for any $i > 0$.

In [3], Hà found certain elements in the inverse images of d_0 and e_0 respectively, and showed that there is no element in the inverse image of g_1 under the $E(2)$-transfer. The discussions in 2.1 and 2.2 explain why Hà's method is no longer applicable to the remaining indecomposable elements f_i, p_i, $D_3(i)$, and p'_i for any i. (It could not directly be applied even to d_i, e_i and g_{i+1} for $i > 0$.)

Using our method, it is not hard to find elements respectively in the inverse images of d_0, e_0, and f_0 under the transfer similarly as we do for p_0.

The contains of this Note will be published in detail elsewhere.

Acknowledgements

The research was in progress during the Fall 2007, when the first named author was visiting to the IHES (Bures-sur-Yvette) and the University Paris 13 (Villetaneuse). He is grateful to Jean Pierre Bourguignon and Lionel Schwartz for organizing his visit. He would like to thank Bob Oliver, Geoffrey Powell, Lionel Schwartz, Micheline Vigué for their warm hospitality and for the lovely working atmosphere.

References

[1] J.M. Boardman, Modular representations on the homology of powers of real projective space, in: Algebraic Topology: Oaxtepec 1991, in: Contemp. Math., vol. 146, Amer Math. Soc., Providence, RI, 1993, pp. 49–70.

[2] R.R. Bruner, L.M. Hà, N.H.V. Hưng, On behavior of the algebraic transfer, Trans. Amer. Math. Soc. 357 (2005) 473–487.

[3] L.M. Hà, Sub-Hopf algebras of the Steenrod algebra and the Singer transfer, in: Proceedings of the School and Conference in Algebraic Topology, Geom. Topol. Publ. Conventry 11 (2007) 81–105.

[4] N.H.V. Hưng, The weak conjecture on spherical classes, Math. Z. 231 (1999) 727–743.

[5] N.H.V. Hưng, The cohomology of the Steenrod algebra and representations of the general linear groups, Trans. Amer. Math. Soc. 357 (2005) 4065–4089.

[6] D.S. Kahn, S.B. Priddy, The transfer and stable homotopy theory, Math. Proc. Cambridge Philos. Soc. 83 (1978) 103–111.

[7] M. Kameko, Products of projective spaces as Steenrod modules, Thesis, Johns Hopkins University, 1990.

[8] W.H. Lin, M. Mahowald, The Adams spectral sequence for Minami's theorem, in: Homotopy Theory via Algebraic Geometry and Group Representations, Evanston, II, 1997, in: Contemp. Math., vol. 220, Amer Math. Soc., Providence, RI, 1998, pp. 143–177.

[9] A. Liulevicius, The factorization of cyclic reduced powers by secondary cohomology operations, Mem. Amer. Math. Soc. 42 (1962).

[10] J.P. May, A General Algebraic Approach to Steenrod Operations, Lecture Notes in Math., vol. 168, Springer-Verlag, 1970, pp. 153–231.

[11] N. Minami, The iterated transfer analogue of the new doomsday conjecture, Trans. Amer. Math. Soc. 351 (1999) 2325–2351.

[12] T.N. Nam, Transfert algébrique et représentation modulaire du groupe linéare, Ann. Inst. Fourier 58 (2008) 1785–1837.

[13] J.H. Palmieri, Quillen stratification for the Steenrod algebra, Ann. of Math. 149 (1999) 421–449.

[14] W.M. Singer, The transfer in homological algebra, Math. Z. 202 (1989) 493–523.

[15] N. Sum, The hit problem for the polynomial algebra of four variables, preprint, 2007, 240 pp.

[16] M.C. Tangora, On the cohomology of the Steenrod algebra, Math. Z. 116 (1970) 18–64.

[17] A. Zachariou, A polynomial subalgebra of the cohomology of the Steenrod algebra, Publ. Res. Inst. Math. Sci. 9 (1973/74) 157–164.