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COMPTES RENDUS MATHEMATIQUE

Probability Theory

Viability property on Riemannian manifolds

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Abstract

This Note studies a sufficient and necessary condition for the viability property of a state system in a closed subset K of a finite-dimensional compact Riemannian manifold without boundary. Our result is: the system enjoys the viability property in K if and only if the square of the distance function of K is a viscosity supersolution of a second-order partial differential equation in some neighborhood of K. ***To cite this article: S. Peng, X. Zhu, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Propriété de viabilité sur une variété riemannienne. Dans cette Note on donne une condition nécessaire et suffisante pour que soit satisfaite la propriété de viabilité d'un système sur un sous-ensemble K d'une variété riemannienne, de dimension finie, sans bord. Le résultat s'énonce ainsi : le système sur K possède la propriété de viabilité si et seulement si le carré de la fonction distance à K est une sursolution de viscosité d'une équation aux dérivées partielles du second ordre définie sur un voisinage de K. ***Pour citer cet article : S. Peng, X. Zhu, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Version française abrégée

Soient $(\Omega, \mathcal{F}, P)$ un espace de probabilité complet et $(W(t), t \geqslant 0)$ un mouvement brownien standard sur cet espace à valeurs dans $\mathbb{R}^d$. Soit $(\mathcal{F}_t)_{t\geqslant 0}$ la filtration naturelle complète engendrée par $(W(t), t \geqslant 0)$ et augmentée des ensembles de P-mesure nulle de $\mathcal{F}$.

Soit M un variété riemannienne complète et sans bord. Sur M on considère dans l'intervalle $[t, T]$, l'équation différentielle stochastique suivante :

$$\begin{cases} dX_s^{t,x} = V_0(s, X_s^{t,x})\,ds + V_\alpha(s, X_s^{t,x}) \circ dW_s^\alpha, \\ X_t^{t,x} = x \in M, \end{cases} \tag{1}$$

où $V_0, V_1, \ldots, V_d$ sont $d + 1$ champs vectoriels réguliers sur M paramétrés par $s \in [0, T]$.

D'après [3], il existe un unique processus, à valeurs dans M, qui résout l'équation (1).

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Soit $i_M(x)$ le rayon d'injectivité de M en x. On a :

$$i_M := \inf\{i_M(x)\colon x \in M\} > 0.$$

La distance de x à un ensemble $K \subset M$ est notée par $d_K(x) := \inf_{y\in K} d(x,y)$, où $d(\cdot,\cdot)$ est la distance riemannienne de M. Soit,

$$\Gamma_\varepsilon = \{y \in M \mid d_K(y) < \varepsilon\},$$

et

$$\tau = \inf\{s \geqslant t\colon d_K(X_s^{t,x}) = \varepsilon\},$$

où $\varepsilon = \frac{i_M}{6}$.

Soit $\tilde{V}_\alpha$ le champ vectoriel sur $M \times M$ défini par :

$$\forall (t,x,y) \in [0,+\infty) \times M \times M, \quad \tilde{V}_\alpha(t,x,y) = (V_\alpha(t,x), V_\alpha(t,y)).$$

Nos hypothèses sont les suivantes :

pour tout couple $x, y \in \Gamma_\varepsilon$, on a $d(x,y) < \min\{i_M(x), i_M(y)\}$, $t \in [0,+\infty)$,

(H1) $\|L_{xy}V_0(t,x) - V_0(t,y)\| \leqslant \mu d(x,y)$,
(H2) $\sum_{\alpha=1}^d \langle D^2(d^2(x,y))\tilde{V}_\alpha(t,x,y), \tilde{V}_\alpha(t,x,y)\rangle \leqslant 2C_0 d^2(x,y)$,
(H3) $\sum_{\alpha=1}^d \|L_{xy}\nabla_{V_\alpha(t,x)}V_\alpha(t,x) - \nabla_{V_\alpha(t,y)}V_\alpha(t,y)\| \leqslant C_1^* d(x,y)$,
(H4) $\sum_{\alpha=1}^d \|L_{xy}V_\alpha(t,x) - V_\alpha(t,y)\| \leqslant C_2^* d(x,y)$,

où C_0, μ, C_1^* et C_2^* sont des constantes positives telles que $C := 2\mu + C_0 + C_1^* + 1 > 0$.

Etant donné K un ensemble fermé de M. Nous considérons la fonction suivante :

$$U(t,x) = E\left[\int_t^{T\wedge\tau} e^{-C(s-t)} d_K^2(X_s^{t,x})\,ds + e^{-C(T\wedge\tau-t)} d_K^2(X_{T\wedge\tau}^{t,x})\right], \quad (t,x) \in [0,T] \times overline\Gamma_\varepsilon.$$

D'après [4], $U(t,x)$ est l'unique solution de viscosité de l'équation de Hamilton–Jacobi :

$$\begin{cases} u_t(t,x) + (V_0 u)(t,x) + \frac{1}{2}\sum_{\alpha=1}^d (V_\alpha V_\alpha u)(t,x) + d_K^2(x) - Cu(t,x) = 0, & (t,x) \in (0,T) \times \Gamma_\varepsilon, \\ u(t,x) = d_K^2(x), & (t,x) \in (0,T] \times \partial\Gamma_\varepsilon, \\ u(T,x) = d_K^2(x), & x \in \overline{\Gamma}_\varepsilon. \end{cases} \tag{2}$$

Théorème 0.1. *Sous les hypothèses* (H1)–(H4)*, les assertions suivantes sont équivalentes* :

(i) *K est viable pour* (1)[1] ;
(ii) *$d_K^2(x)$ est une sursolution de viscosité de* (2).[2]

1. Introduction

Let $(W(t),\ t \geqslant 0)$ be a d-dimensional standard Brownian motion on some complete probability space $(\Omega, \mathcal{F}, P)$. We denote by $(\mathcal{F}_t)_{t\geqslant 0}$ the natural filtration generated by W and augmented by the P-null sets of $\mathcal{F}$.

In the sequel, we assume that M is a compact Riemannian manifold without boundary. We consider the following stochastic differential equation on M in a fixed time interval $[t,T]$:

$$\begin{cases} dX_s^{t,x} = V_0(s, X_s^{t,x})\,ds + V_\alpha(s, X_s^{t,x}) \circ dW_s^\alpha, \\ X_t^{t,x} = x \in M, \end{cases} \tag{3}$$

where $V_0, V_1, \ldots, V_d$ are $d+1$ one-parameter smooth vector fields on M.

[1] On dit que K est viable si, quitte à changer d'espace de probabilié, pour tout $(t,x) \in [0,T] \times K$, alors, P-p.s. $X_s^{t,x} \in K$, $\forall s \in [t,T]$.
[2] Pour la définition d'une sursolution de viscosité, voir [1].

Since M is compact and without boundary, according to [3], there exists a unique M-valued continuous process which solves Eq. (3). Moreover, this solution does not blow up.

Let K be a given closed subset of M. The aim of this paper is to give a necessary and sufficient condition that keeps the state $(X_s^{t,x},\ t \leqslant s \leqslant T)$ in K whenever $x \in K$.

Definition 1.1. We say that K enjoys the viability property with respect to (1) if, for all $(t,x) \in [0,T] \times K$, there exist a probability space $(\Omega, \mathcal{F}, P)$, a d-dimensional Brownian motion W, such that, P-a.s., $X_s^{t,x} \in K$, $\forall s \in [t,T]$.

The case $M = R^n$ has been studied in [2] for time independent coefficients. However, general manifolds have not been considered in [2]. Our approach is similar to that in [2].

Let $i_M(x)$ denote the injectivity radius of M at x. Since M is compact, we know

$$i_M := \inf\{i_M(x):\ x \in M\} > 0.$$

We denote by $d_K(x) := \inf_{y \in K} d(x,y)$ the distance from x to K, where $d(\cdot,\cdot)$ is the Riemannian distance in M. Set,

$$\Gamma_\varepsilon = \{y \in M \mid d_K(y) < \varepsilon\}$$

and

$$\tau = \inf\{s \geqslant t:\ d_K(X_s^{t,x}) = \varepsilon\},$$

where $\varepsilon = \frac{i_M}{6}$. Then our problem relates with the following function:

$$U(t,x) = E\left[\int_t^{T\wedge\tau} e^{-C(s-t)} d_K^2(X_s^{t,x})\,ds + e^{-C(T\wedge\tau - t)} d_K^2(X_{T\wedge\tau}^{t,x})\right], \quad (t,x) \in [0,T] \times \overline{\Gamma}_\varepsilon. \tag{4}$$

The constant C in (4) introduced later depends on the $d+1$ one-parameter tangent vector fields in (3).

We denote by L_{xy} the parallel transport along the unique minimizing geodesic connecting x to y. Let $\tilde{V}_\alpha$ denote the one-parameter tangent vector field on $M \times M$ which is defined as follows:

$$\forall (t,x,y) \in [0,+\infty) \times M \times M, \quad \tilde{V}_\alpha(t,x,y) = (V_\alpha(t,x), V_\alpha(t,y)). \tag{5}$$

We assume that, for all $x, y \in \Gamma_\varepsilon$ s.t. $d(x,y) < \min\{i_M(x), i_M(y)\}$, $t \in [0,+\infty)$,

(H1) $\|L_{xy}V_0(t,x) - V_0(t,y)\| \leqslant \mu d(x,y)$,
(H2) $\sum_{\alpha=1}^d \langle D^2(d^2(x,y))\tilde{V}_\alpha(t,x,y), \tilde{V}_\alpha(t,x,y)\rangle \leqslant 2C_0 d^2(x,y)$,
(H3) $\sum_{\alpha=1}^d \|L_{xy}\nabla_{V_\alpha(t,x)}V_\alpha(t,x) - \nabla_{V_\alpha(t,y)}V_\alpha(t,y)\| \leqslant C_1^* d(x,y)$,
(H4) $\sum_{\alpha=1}^d \|L_{xy}V_\alpha(t,x) - V_\alpha(t,y)\| \leqslant C_2^* d(x,y)$,

where C_0, μ, C_1^* and C_2^* are positive constants, and $C := 2\mu + C_0 + C_1^* + 1 > 0$.

Consider now the following Hamilton–Jacobi equation:

$$\begin{cases} u_t(t,x) + (V_0 u)(t,x) + \frac{1}{2}\sum_{\alpha=1}^d (V_\alpha V_\alpha u)(t,x) + d_K^2(x) - Cu(t,x) = 0, & (t,x) \in (0,T) \times \Gamma_\varepsilon, \\ u(t,x) = d_K^2(x), & (t,x) \in (0,T] \times \partial\Gamma_\varepsilon, \\ u(T,x) = d_K^2(x), & x \in \overline{\Gamma}_\varepsilon. \end{cases} \tag{6}$$

Definition 1.2. (See [1].) A continuous function $u : [0,T] \times \overline{\Gamma}_\varepsilon \to R$ is a viscosity supersolution (subsolution) of (6) if $u(t,x) \geqslant (\leqslant) d_K^2(x)$ for all $(t,x) \in (0,T] \times \partial\Gamma_\varepsilon$ and $u(T,x) \geqslant (\leqslant) d_K^2(x)$, for all $x \in \overline{\Gamma}_\varepsilon$, and for any $\varphi \in C^{1,2}((0,T) \times \Gamma_\varepsilon)$ and any $(t,x) \in (0,T) \times \Gamma_\varepsilon$ at which $u - \varphi$ attains a local minimum (maximum),

$$\varphi_t(t,x) + (V_0\varphi)(t,x) + \frac{1}{2}\sum_{\alpha=1}^d (V_\alpha V_\alpha \varphi)(t,x) + d_K^2(x) - Cu(t,x) \leqslant (\geqslant) 0.$$

If u is both a viscosity subsolution and a viscosity supersolution of (6), we say that u is a viscosity solution of (6).

We will see in the next section that $U(t,x)$ is the unique viscosity solution of (6). The last section is devoted to the study of a sufficient and necessary condition for the viability property of K with respect to (3).

2. The unique viscosity solution to the Hamilton–Jacobi equations

For completeness, we first recall the comparison theorem of viscosity solution to the following parabolic PDE on Riemannian manifolds:

$$\begin{cases} u_t + F(t,x,u,du,d^2u) = 0 & \text{in } (0,T)\times\Gamma_\varepsilon, \\ u(t,x) = f(t,x), & (t,x)\in[0,T)\times\partial\Gamma_\varepsilon, \\ u(0,x) = \psi(x), & x\in\overline{\Gamma}_\varepsilon, \end{cases} \tag{7}$$

where du, d^2u mean $d_xu(t,x)$ and $d_x^2u(t,x)$ (see [1]). $T>0$, $\psi\in C(\overline{\Gamma}_\varepsilon)$ and $f\in C([0,T)\times\partial\Gamma_\varepsilon)$ are given.

We denote by TM_x^* the cotangent space of M at a point $x\in M$. Let TM_x stand for the tangent space at x and $\mathcal{L}_s^2(TM_x)$ denote the symmetric bilinear forms on TM_x. Set,

$$\chi := \big\{(t,x,r,\zeta,A)\colon t\in[0,T],\ x\in M,\ r\in R,\ \zeta\in TM_x^*,\ A\in\mathcal{L}_s^2(TM_x)\big\}.$$

Proposition 2.1. *(See [4].) Let Γ_ε be an open subset of a compact finite-dimensional Riemannian manifold M, and $F\in C(\chi,R)$ be continuous, proper for each fixed $t\in(0,T)$ and satisfy: there exists a function $\omega:[0,+\infty]\to[0,+\infty]$ with $\omega(0+)=0$ and such that*

$$F\big(t,y,r,\alpha\exp_y^{-1}(x),Q\big) - F\big(t,x,r,-\alpha\exp_x^{-1}(y),P\big) \leqslant \omega\big(\alpha d^2(x,y)+d(x,y)\big), \tag{8}$$

for all fixed $t\in(0,T)$ and for all $x,y\in\Gamma_\varepsilon$ which satisfy $d(x,y)<\min\{i_M(x),i_M(y)\}$, $r\in R$, $P\in\mathcal{L}_s^2(TM_x)$, $Q\in\mathcal{L}_s^2(TM_y)$ with

$$-\left(\frac{1}{\varepsilon_\alpha}+\|A_\alpha\|\right)\begin{pmatrix} I & 0\\ 0 & I\end{pmatrix} \leqslant \begin{pmatrix} P & 0\\ 0 & -Q\end{pmatrix} \leqslant A_\alpha + \varepsilon_\alpha A_\alpha^2, \tag{9}$$

where A_α is the second derivative of the function $\varphi_\alpha(x,y)=\frac{\alpha}{2}d^2(x,y)$ $(\alpha>0)$ at the point $(x,y)\in M\times M$, and

$$\varepsilon_\alpha = \frac{1}{2(1+\|A_\alpha\|)}.$$

If u (resp., v) is an upper (resp., lower) semicontinuous function on $[0,T)\times\overline{\Gamma}_\varepsilon$ and a subsolution (resp., supersolution) of (7) then $u\leqslant v$ on $[0,T)\times\Gamma_\varepsilon$.

In particular PDE (7) has at most one viscosity solution.

Proposition 2.2. *The function $U(t,x)$ introduced in* (4) *is bounded and continuous in* $[0,T]\times\overline{\Gamma}_\varepsilon$. *Moreover, it's the unique viscosity solution of* (6).

Proof. The function $U(t,x)$ is bounded since M is compact. For the proof of the continuity of $U(t,x)$, see our forthcoming paper.

If we note that

$$U(t,x) = E\left[\int_t^{(t+\delta)\wedge\tau} e^{-C(s-t)}d_K^2\big(X_s^{t,x}\big)\,ds + e^{-C[(t+\delta)\wedge\tau-t]}U\big((t+\delta)\wedge\tau, X_{(t+\delta)\wedge\tau}^{t,x}\big)\right],$$

we can use Definition 1.2 to prove, with the help of Itô's formula that $U(t,x)$ is a viscosity solution of (6).

Then we will use Proposition 2.1 to get the uniqueness result. After transforming our PDE (6) to the standard form in (7), our function $F:\chi\to R$ is as follows:

$$F(t,x,r,\zeta,P) = -d_K^2(x) + Cr - \big\langle\zeta, V_0(t,x)\big\rangle - \frac{1}{2}\sum_{\alpha=1}^{d}\big\langle\zeta, \nabla_{V_\alpha(t,x)}V_\alpha(t,x)\big\rangle - \frac{1}{2}\sum_{\alpha=1}^{d}\big\langle PV_\alpha(t,x), V_\alpha(t,x)\big\rangle. \tag{10}$$

Since M is compact, there exists a constant $k_0>0$ s.t. the sectional curvature is bounded below by $-k_0$ on M. Then, according to [1], (9) implies:

$$P - L_{yx}(Q) \leqslant \frac{3}{2}k_0\alpha d^2(x,y)I.$$

With the help of this relation and (H1)–(H4), we can easily prove that there exists a function $\omega : [0, +\infty] \to [0, +\infty]$ with $\omega(0+) = 0$ such that (8) holds true for our function F introduced in (10). So by Proposition 2.1, we can get the uniqueness result. □

3. Characterization of the viability property

Theorem 3.1. *Under the assumptions* (H1)–(H4), *the following conditions are equivalent*:

(i) K *enjoys viability with respect to* (3);
(ii) $d_K^2(x)$ *is a viscosity supersolution of* (6).

Proof. (ii) ⇒ (i): If $d_K^2(x)$ is a viscosity supersolution of (6), then, by Propositions 2.1 and 2.2, one has $U(t,x) \leqslant d_K^2(x), \forall (t,x) \in (0,T] \times \Gamma_\varepsilon$. Thus, in particular, for all $(t,x) \in (0,T] \times K$, $U(t,x) = 0$, what is equivalent to: for all $t \in [0,T]$,

$$X_s^{t,x} \in K, \quad \forall s \in [t,T], \ P\text{-a.s.}$$

(i) ⇒ (ii): For any given $(t,x) \in (0,T) \times \Gamma_\varepsilon$, let $\bar{x} \in K$ such that $d_K^2(x) = d^2(x,\bar{x})$ (if $x \in K, x = \bar{x}$). Suppose that K enjoys the viability property for (1). Then we have that, a.s., $X_s^{t,\bar{x}} \in K, \forall s \in [t,T]$. Let $\varphi \in C^{1,2}((0,T) \times \Gamma_\varepsilon)$ satisfies:

$$d_K^2(x) - \varphi(t,x) = 0 \leqslant d_K^2(x') - \varphi(t',x'), \quad \text{for } (t',x') \text{ in a neighborhood of } (t,x). \tag{11}$$

For all $0 < \eta < \varepsilon$, we define:

$$\tau_\eta = \eta \wedge \inf\{s \geqslant 0,\ d(x, X_{t+s}^{t,x}) > \eta\} \wedge \inf\{s \geqslant 0,\ d(\bar{x}, X_{t+s}^{t,\bar{x}}) > \eta\}.$$

By (9), for $\eta > 0$ small enough, we have:

$$\varphi(t+\tau_\eta, X_{t+\tau_\eta}^{t,x}) - \varphi(t,x) \leqslant d_K^2(X_{t+\tau_\eta}^{t,x}) - d_K^2(x). \tag{12}$$

Applying Itô's formula to the left-hand side of this inequality, we get

$$E[\varphi(t+\tau_\eta, X_{t+\tau_\eta}^{t,x}) - \varphi(t,x)] = E\int_t^{t+\tau_\eta} \left[\varphi_s(s, X_s^{t,x}) + (V_0\varphi)(s, X_s^{t,x}) + \frac{1}{2}\sum_{\alpha=1}^d (V_\alpha V_\alpha \varphi)(s, X_s^x)\right] ds.$$

With the help of this relation we can prove easily that

$$\lim_{\eta\to 0} \frac{E[\varphi(t+\tau_\eta, X_{t+\tau_\eta}^{t,x}) - \varphi(t,x)]}{E\tau_\eta} = \varphi_t(t,x) + (V_0\varphi)(t,x) + \frac{1}{2}\sum_{\alpha=1}^d (V_\alpha V_\alpha\varphi)(t,x). \tag{13}$$

Let us now consider the right-hand side of (12). By the viability assumption, we have $X_{t+\tau_\eta}^{t,\bar{x}} \in K$. Thus $d_K^2(X_{t+\tau_\eta}^{t,x}) \leqslant d^2(X_{t+\tau_\eta}^{t,x}, X_{t+\tau_\eta}^{t,\bar{x}})$. On the other hand, for all $s \in [0,\tau_\eta]$,

$$d(X_{t+s}^{t,x}, X_{t+s}^{t,\bar{x}}) \leqslant d(X_{t+s}^{t,x}, x) + d(x,\bar{x}) + d(\bar{x}, X_{t+s}^{t,\bar{x}}) < 3\varepsilon = \frac{i_M}{2}.$$

Since $d^2(x,y)$ is C^∞ smooth on the set $\{(x,y) \in M \times M\colon d(x,y) < \frac{i_M}{2}\}$, we can apply Itô's formula to $d^2(X_s^{t,x}, X_s^{t,\bar{x}}),\ s \in [t, t+\tau_\eta]$,

$$\begin{aligned} &E[d^2(X_{t+\tau_\eta}^{t,x}, X_{t+\tau_\eta}^{t,\bar{x}}) - d^2(x,\bar{x})] \\ &\quad = E\int_t^{t+\tau_\eta} \left[\tilde{V}_0(s, X_s^{t,x}, X_s^{t,\bar{x}}) d^2(X_s^{t,x}, X_s^{t,\bar{x}}) + \frac{1}{2}\sum_{\alpha=1}^d (\tilde{V}_\alpha \tilde{V}_\alpha)(s, X_s^{t,x}, X_s^{t,\bar{x}}) d^2(X_s^{t,x}, X_s^{t,\bar{x}})\right] ds, \end{aligned} \tag{14}$$

where the tangent vector field $\tilde{V}_\alpha$ on $M \times M$ has been defined in (5).

According to [1],

$$\frac{\partial}{\partial x}d^2(x,y) = -2\exp_x^{-1}(y), \qquad \frac{\partial}{\partial y}d^2(x,y) = -2\exp_y^{-1}(x), \qquad \frac{\partial}{\partial x}d^2(x,y) + L_{yx}\frac{\partial}{\partial y}d^2(x,y) = 0.$$

Thus from (H1), we have:

$$\begin{aligned}
&\tilde{V}_0\big(s, X_s^{t,x}, X_s^{t,\bar{x}}\big)d^2\big(X_s^{t,x}, X_s^{t,\bar{x}}\big)\\
&\quad = \big\langle V_0\big(s, X_s^{t,x}\big), -2\exp^{-1}_{X_s^{t,x}}\big(X_s^{t,\bar{x}}\big)\big\rangle + \big\langle V_0\big(s, X_s^{t,\bar{x}}\big), -2\exp^{-1}_{X_s^{t,\bar{x}}}\big(X_s^{t,x}\big)\big\rangle\\
&\quad = \big\langle V_0\big(s, X_s^{t,x}\big), 2L_{X_s^{t,\bar{x}}X_s^{t,x}}\exp^{-1}_{X_s^{t,\bar{x}}}\big(X_s^{t,x}\big)\big\rangle + \big\langle V_0\big(s, X_s^{t,\bar{x}}\big), -2\exp^{-1}_{X_s^{t,\bar{x}}}\big(X_s^{t,x}\big)\big\rangle\\
&\quad = 2\big\langle L_{X_s^{t,x}X_s^{t,\bar{x}}}V_0\big(s, X_s^{t,x}\big) - V_0\big(s, X_s^{t,\bar{x}}\big), \exp^{-1}_{X_s^{t,\bar{x}}}\big(X_s^{t,x}\big)\big\rangle\\
&\quad \leqslant 2\mu d^2\big(X_s^{t,x}, X_s^{t,\bar{x}}\big),
\end{aligned}$$

and combining this estimate with (H2), (H3) and (14), we get:

$$\begin{aligned}
&E\big[d^2\big(X_{t+\tau_\eta}^{t,x}, X_{t+\tau_\eta}^{t,\bar{x}}\big) - d^2(x,\bar{x})\big]\\
&\quad \leqslant \big(2\mu + C_0 + C_1^*\big)E\int_t^{t+\tau_\eta} d^2\big(X_s^{t,x}, X_s^{t,\bar{x}}\big)\,ds\\
&\quad \leqslant (C-1)E\int_t^{t+\tau_\eta}\left[\left(1+\frac{1}{\eta}\right)d^2\big(X_s^{t,x}, x\big) + \left(1+\frac{1}{\eta}\right)(1+\eta)d^2\big(X_s^{t,\bar{x}}, \bar{x}\big) + (1+\eta)^2 d^2(x,\bar{x})\right]ds\\
&\quad \leqslant (C-1)(1+\eta)^2 E\tau_\eta d^2(x,\bar{x}) + O(\eta)E\tau_\eta.
\end{aligned}$$

Substitution of the latter estimate and of (13) and (12) yields:

$$\varphi_t(t,x) + (V_0\varphi)(t,x) + \frac{1}{2}\sum_{\alpha=1}^{d}(V_\alpha V_\alpha\varphi)(t,x) \leqslant \lim_{\eta\to 0}\frac{E[d_K^2(X_{t+\tau_\eta}^{t,x}) - d_K^2(x)]}{E\tau_\eta} \leqslant (C-1)d^2(x,\bar{x}).$$

Finally, from $d^2(x,\bar{x}) = d_K^2(x) = \varphi(t,x)$ we deduce that

$$\varphi_t(t,x) + (V_0\varphi)(t,x) + \frac{1}{2}\sum_{\alpha=1}^{d}(V_\alpha V_\alpha\varphi)(t,x) + d_K^2(x) - C\varphi(t,x) \leqslant 0. \qquad \square$$

Remark. Our assumptions are just the analogue to the Lipschitz conditions for the coefficients of SDEs in R^n. Our result extends that of [2] to SDEs on Riemannian manifolds.

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