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A minimum degree condition of fractional (k, m)-deleted graphs ☆

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Abstract

Let G be a graph of order n, and let $k \geqslant 1$ and $m \geqslant 1$ be two integers. In this paper, we consider the relationship between the minimum degree $\delta(G)$ and the fractional (k, m)-deleted graphs. It is proved that if $n \geqslant 4k - 5 + 2(2k + 1)m$ and $\delta(G) \geqslant \frac{n}{2}$, then G is a fractional (k, m)-deleted graph. Furthermore, we show that the minimum degree condition is sharp in some sense. ***To cite this article: S. Zhou, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Une condition sur le degré minimal pour qu'un graphe soit (k, m)-effacé fractionnaire. Soit G un graphe d'ordre n et $k \geqslant 1$, $m \geqslant 1$ deux entiers, nous notons $\delta(G)$ le degré minimal de G. Dans cette Note nous montrons que si $n \geqslant 4k - 5 + 2(2k + 1)m$ et $\delta(G) \geqslant n/2$ alors G est un graphe (k, m)-effacé fractionnaire. De plus, nous montrons par un exemple que la condition sur le degré minimal ne peut être remplacée par $\delta(G) \geqslant (n - 1)/2$. ***Pour citer cet article : S. Zhou, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

1. Introduction

The reader is referred to [1] for undefined terms and concepts. We consider finite undirected graphs without loops or multiple edges. Let G be a graph of order n. We use $V(G)$ and $E(G)$ to denote its vertex set and edge set, respectively. For any $x \in V(G)$, the degree of x in G is denoted by $d_G(x)$. We write $N_G(x)$ for the set of vertices adjacent to x in G, and $N_G[x]$ for $N_G(x) \cup \{x\}$. For $S \subseteq V(G)$, we write $d_G(S)$ instead of $\sum_{x \in S} d_G(x)$. We denote by $G[S]$ the subgraph of G induced by S, and $G - S = G[V(G) \setminus S]$. Let S and T be two disjoint vertex subsets of G, we use $e_G(S, T)$ to denote the number of edges with one end in S and the other end in T. If $T = \{x\}$, then we write $e_G(x, S)$ instead of $e_G(T, S)$. We use $\delta(G)$ for the minimum degree of G.

Let $k \geqslant 1$ be an integer. Then a spanning subgraph F of G is called a k-factor if $d_F(x) = k$ for each $x \in V(G)$. Let $h : E(G) \to [0, 1]$ be a function. If $\sum_{e \ni x} h(e) = k$ holds for any $x \in V(G)$, then we call $G[F_h]$ a fractional k-factor

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of G with indicator function h where $F_h = \{e \in E(G)\colon\ h(e) > 0\}$. In this paper we introduce firstly the definition of a fractional (k,m)-deleted graph, that is, a graph G is called a fractional (k,m)-deleted graph if there exists a fractional k-factor $G[F_h]$ of G with indicator function h such that $h(e) = 0$ for any $e \in E(H)$, where H is any subgraph of G with m edges. A fractional (k,m)-deleted graph is simply called a fractional k-deleted graph if $m = 1$.

Many authors have investigated k-factors or fractional k-factors [2,4–6]. The following results on k-factors and fractional k-factors are known:

Theorem 1 *(Katerinis [2]). Let $k \geqslant 1$ be an integer, and let G be a graph of order n with $n \geqslant 4k-5$, kn even. If $\delta(G) \geqslant \frac{n}{2}$, then G has a k-factor.*

Yu showed a degree condition for the existence of a fractional k-factor.

Theorem 2 *(Yu [5]). Let k be an integer with $k \geqslant 1$, and let G be a connected graph of order n with $n \geqslant 4k-3$, $\delta(G) \geqslant k$. If $\max\{d_G(x), d_G(y)\} \geqslant \frac{n}{2}$ for each pair of nonadjacent vertices x, y of G, then G has a fractional k-factor.*

From Theorem 2, we easily get the following result:

Theorem 3. *Let $k \geqslant 1$ be an integer, and let G be a connected graph of order n with $n \geqslant 4k-3$. If $\delta(G) \geqslant \frac{n}{2}$, then G has a fractional k-factor.*

The toughness $t(G)$ of a graph G was defined as follows: $t(G) = \min\{\frac{|S|}{\omega(G-S)}\colon\ S \subseteq V(G),\ \omega(G-S) \geqslant 2\}$, if G is not complete, where $\omega(G-S)$ denotes the number of components of G–S; otherwise, set $t(G) = +\infty$. Liu and Zhang gave a toughness condition for graphs to have fractional k-factors.

Theorem 4 *(Liu and Zhang [4]). Let $k \geqslant 2$ be an integer. A graph G of order n with $n \geqslant k+1$ has a fractional k-factor if $t(G) \geqslant k - \frac{1}{k}$.*

In this paper, we obtain a minimum degree condition for a graph to be a fractional (k,m)-deleted graph. Our result is an extension of Theorems 1 and 3.

Theorem 5. *Let $k \geqslant 1$ and $m \geqslant 1$ be two integers. Let G be a graph of order n with $n \geqslant 4k-5+2(2k+1)m$. If $\delta(G) \geqslant \frac{n}{2}$, then G is a fractional (k,m)-deleted graph.*

In Theorem 5, if $m = 1$, then we get the following corollary:

Corollary 1. *Let $k \geqslant 1$ be an integer. Let G be a graph of order n with $n \geqslant 8k-3$. If $\delta(G) \geqslant \frac{n}{2}$, then G is a fractional k-deleted graph.*

2. The proof of Theorem 5

In order to prove Theorem 5, we depend on the following lemmas:

Lemma 2.1 *(Liu [3]). Let G be a graph. Then G has a fractional k-factor if and only if for every subset S of $V(G)$, $\delta_G(S,T) = k|S| + d_{G-S}(T) - k|T| \geqslant 0$, where $T = \{x\colon\ x \in V(G) \setminus S,\ d_{G-S}(x) \leqslant k-1\}$.*

Lemma 2.2. *Let $k \geqslant 1$ and $m \geqslant 0$ be two integers, and let G be a graph and H a subgraph of G with m edges. Then G is a fractional (k,m)-deleted graph if and only if for any subset S of $V(G)$, $\delta_G(S,T) = k|S| + \sum_{x\in T} d_{G-S}(x) - k|T| \geqslant \sum_{x\in T} d_H(x) - e_H(S,T)$, where $T = \{x\colon\ x \in V(G) \setminus S,\ d_{G-S}(x) - d_H(x) + e_H(x,S) \leqslant k-1\}$.*

Proof. Let $G' = G - E(H)$. Then G is a fractional (k,m)-deleted graph if and only if G' has a fractional k-factor. According to Lemma 2.1, this is true if and only if for any subset S of $V(G)$, $\delta_{G'}(S,T') = k|S| + d_{G'-S}(T') - k|T'| \geqslant 0$, where $T' = \{x\colon\ x \in V(G) \setminus S,\ d_{G'-S}(x) \leqslant k-1\}$.

It is easy to see that $d_{G'-S}(x) = d_{G-S}(x) - d_H(x) + e_H(x,S)$ for any $x \in T'$. By the definitions of T' and T, we have $T' = T$. Hence, we obtain $\delta_{G'}(S,T') = \delta_G(S,T) - \sum_{x\in T} d_H(x) + e_H(S,T)$. Thus, $\delta_{G'}(S,T') \geqslant 0$ if and only if $\delta_G(S,T) \geqslant \sum_{x\in T} d_H(x) - e_H(S,T)$. It follows that G is a fractional (k,m)-deleted graph if and only if $\delta_G(S,T) = k|S| + \sum_{x\in T} d_{G-S}(x) - k|T| \geqslant \sum_{x\in T} d_H(x) - e_H(S,T)$. □

Proof of Theorem 5. Suppose that G satisfies the assumption of the theorem, but is not a fractional (k,m)-deleted graph. Then by Lemma 2.2, there exists some subset S of $V(G)$ such that

$$k|S| + \sum_{x\in T}\big(d_{G-S}(x) - d_H(x) + e_H(x,S) - k\big) \leqslant -1, \tag{1}$$

where $T = \{x\colon\ x \in V(G)\setminus S,\ d_{G-S}(x) - d_H(x) + e_H(x,S) \leqslant k-1\}$.

At first, we prove the following claims:

Claim 1. $|S| \geqslant 1$.

Proof. If $S = \emptyset$, then by (1), we have $-1 \geqslant \sum_{x\in T}(d_G(x) - d_H(x) - k) \geqslant \sum_{x\in T}(\delta(G) - m - k) \geqslant 0$, this is a contradiction. □

Claim 2. $|T| \geqslant k+1$.

Proof. If $|T| \leqslant k$, then by (1), Claim 1 and $\delta(G) \geqslant \frac{n}{2}$, we have

$$\begin{aligned}
-1 &\geqslant k|S| + \sum_{x\in T}\big(d_{G-S}(x) - d_H(x) + e_H(x,S) - k\big) \geqslant |T||S| + \sum_{x\in T}\big(d_{G-S}(x) - d_H(x) + e_H(x,S) - k\big)\\
&= \sum_{x\in T}\big(|S| + d_{G-S}(x) - d_H(x) + e_H(x,S) - k\big) \geqslant \sum_{x\in T}\big(d_G(x) - d_H(x) + e_H(x,S) - k\big)\\
&\geqslant \sum_{x\in T}\big(\delta(G) - m - k\big) \geqslant 0,
\end{aligned}$$

which is a contradiction. □

According to Claim 2, we have $T \neq \emptyset$. Thus, we may define $h = \min\{d_{G-S}(x) - d_H(x) + e_H(x,S) \mid x \in T\}$. And let x_1 be a vertex in T satisfying $d_{G-S}(x_1) - d_H(x_1) + e_H(x_1,S) = h$. Then we have $0 \leqslant h \leqslant k-1$ according to the definition of T and $d_G(x_1) \leqslant d_{G-S}(x_1) + |S| = h + d_H(x_1) - e_H(x_1,S) + |S|$.

In view of the condition of Theorem 5, the following inequalities hold:

$$\begin{aligned}
&\tfrac{n}{2} \leqslant \delta(G) \leqslant d_G(x_1) \leqslant h + d_H(x_1) - e_H(x_1,S) + |S|, \quad \text{that is,}\\
&|S| \geqslant \tfrac{n}{2} - \big(h + d_H(x_1) - e_H(x_1,S)\big).
\end{aligned} \tag{2}$$

Now in order to prove the theorem, we shall deduce some contradictions in view of the following two cases:

Case 1. $h = 0$.

By (1), (2) and $|S| + |T| \leqslant n$, we get

$$\begin{aligned}
-1 &\geqslant k|S| + \sum_{x\in T}\big(d_{G-S}(x) - d_H(x) + e_H(x,S) - k\big) \geqslant k|S| + h|T| - k|T| = k|S| - k|T|\\
&\geqslant k|S| - k\big(n - |S|\big) = 2k|S| - kn \geqslant 2k\big(\tfrac{n}{2} - \big(d_H(x_1) - e_H(x_1,S)\big)\big) - kn = -2k\big(d_H(x_1) - e_H(x_1,S)\big),
\end{aligned}$$

which implies $d_H(x_1) - e_H(x_1,S) \geqslant \frac{1}{2k} > 0$.

According to the integrality of $d_H(x_1) - e_H(x_1,S)$, we have $d_H(x_1) - e_H(x_1,S) \geqslant 1$.

For some $x \in T \setminus \{x_1\}$, if $d_{G-S}(x) - d_H(x) + e_H(x,S) = 0$, then we similarly get $d_H(x) - e_H(x,S) \geqslant 1$. Hence, one of (a) and (b) holds for any $x \in T \setminus \{x_1\}$:

(a) $d_{G-S}(x) - d_H(x) + e_H(x,S) \geqslant 1$ or (b) $d_{G-S}(x) - d_H(x) + e_H(x,S) = 0$ and $d_H(x) - e_H(x,S) \geqslant 1$.

Thus, we have

$$\sum_{x \in T} \big(d_{G-S}(x) - d_H(x) + e_H(x,S)\big) \geqslant |T| - 2m. \tag{3}$$

In view of (1), (2), (3), $h = 0$, $|S| + |T| \leqslant n$ and $n \geqslant 4k - 5 + 2(2k+1)m$, we get

$$\begin{aligned} -1 &\geqslant k|S| + \sum_{x \in T} \big(d_{G-S}(x) - d_H(x) + e_H(x,S) - k\big) \geqslant k|S| + |T| - 2m - k|T| \\ &= k|S| - (k-1)|T| - 2m \geqslant k|S| - (k-1)\big(n - |S|\big) - 2m = (2k-1)|S| - (k-1)n - 2m \\ &\geqslant (2k-1)\big(\tfrac{n}{2} - \big(d_H(x_1) - e_H(x_1,S)\big)\big) - (k-1)n - 2m \geqslant (2k-1)\big(\tfrac{n}{2} - m\big) - (k-1)n - 2m \\ &= \tfrac{n}{2} - (2k+1)m \geqslant \tfrac{4k-5+2(2k+1)m}{2} - (2k+1)m > 2k - 3 \geqslant -1, \end{aligned}$$

a contradiction.

Case 2. $1 \leqslant h \leqslant k - 1$.

According to (1), (2), $n \geqslant 4k - 5 + 2(2k+1)m$ and $|S| + |T| \leqslant n$, we obtain

$$\begin{aligned} -1 &\geqslant k|S| + \sum_{x \in T} \big(d_{G-S}(x) - d_H(x) + e_H(x,S) - k\big) \geqslant k|S| + h|T| - k|T| \\ &= k|S| - (k-h)|T| \geqslant k|S| - (k-h)\big(n - |S|\big) = (2k-h)|S| - (k-h)n \\ &\geqslant (2k-h)\big(\tfrac{n}{2} - \big(h + d_H(x_1) - e_H(x_1,S)\big)\big) - (k-h)n \\ &= \tfrac{hn}{2} - (2k-h)\big(h + d_H(x_1) - e_H(x_1,S)\big) \geqslant \tfrac{hn}{2} - (2k-h)(h+m) \\ &\geqslant \tfrac{h(4k-5+2(2k+1)m)}{2} - (2k-h)(h+m) \\ &> \tfrac{h(4k-6+2(2k+1)m)}{2} - (2k-h)(h+m) = h^2 + 2(k+1)mh - 3h - 2km, \end{aligned}$$

that is,

$$-1 > h^2 + 2(k+1)mh - 3h - 2km. \tag{4}$$

Let $f(h) = h^2 + 2(k+1)mh - 3h - 2km$. Clearly, the function $f(h)$ attains its minimum value at $h = 1$ since $1 \leqslant h \leqslant k - 1$. Then we get $f(h) \geqslant f(1)$. Combining this with (4) and $m \geqslant 1$, we have $-1 > f(h) \geqslant f(1) = 2m - 2 \geqslant 0$. It is a contradiction.

Completing the proof of Theorem 5. □

Remark. Let us show that the condition $\delta(G) \geqslant \frac{n}{2}$ in Theorem 5 cannot be replaced by $\delta(G) \geqslant \frac{n-1}{2}$. Let $G = K_{2k-3+(2k+1)m} \bigvee (((2k-1)m + 2k - 2)K_1 \cup (mK_2))$. Then we have $n = 4k - 5 + 2(2k+1)m$ and $\delta(G) = 2k - 3 + (2k+1)m = \frac{n-1}{2}$. Let $G' = G - E(mK_2)$, $S = V(K_{2k-3+(2k+1)m}) \subseteq V(G)$ and $T = V(((2k-1)m + 2k - 2)K_1 \cup (mK_2)) \subseteq V(G)$, then $|S| = 2k - 3 + (2k+1)m$, $|T| = 2k - 2 + (2k+1)m$ and $d_{G'-S}(T) = 0$. Thus, we get $\delta_{G'}(S,T) = k|S| + d_{G'-S}(T) - k|T| = k(2k - 3 + (2k+1)m) - k(2k - 2 + (2k+1)m) = -k < 0$. By Lemma 2.1, G' has no fractional k-factor. Hence, G is not a fractional (k,m)-deleted graph. In the above sense, the result in Theorem 5 is best possible.

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