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Partial Differential Equations/Optimal Control

On the local controllability of a class of multidimensional quasilinear parabolic equations

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Abstract

In this Note, we study the local null controllability and the cost estimate for a class of multidimensional quasilinear parabolic equations with homogeneous Dirichlet boundary conditions and an arbitrary located internal controller. Unlike the known result for one space dimension, we need to consider the problem in the frame of classical solutions. The key point is to improve the regularity of control function for smooth data, which is a consequence of a new observability inequality for linear parabolic equations with an explicit estimate on the observability constant in terms of the C^1-norm of the principle part coefficients. The later is based on a new global Carleman estimate for the linear parabolic equation. ***To cite this article: X. Liu, X. Zhang, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Contrôlabilité locale d'une classe d'équations quasi-linéaires paraboliques multidimensionnelles. Dans cette Note, nous étudions la contrôlabilité locale vers zéro et son coût pour une classe d'équations quasi-linéaires paraboliques multidimensionnelles avec une condition homogène de Dirichlet et un contrôle interne. À la différence des résultats connus dans le cas monodimensionel, nous avons besoin de considérer le problème dans le cadre des solutions classiques. Le point clé consiste à améliorer la régularité de la fonction contrôle pour des données régulières. Ceci découle d'une nouvelle inégalité d'observabilité pour les équations linéaires paraboliques dans laquelle la constante d'observabilité est explicite vis-à-vis de la norme C^1 des coefficients de la partie principale. À cette fin, on établit une nouvelle inégalité de Carleman globale pour les équations linéaires paraboliques. ***Pour citer cet article : X. Liu, X. Zhang, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Version française abrégée

Soit Ω un domaine borné de $\mathbb{R}^n$ avec une frontère Γ dans C^3. Posons $Q = \Omega \times (0, T)$ et $\Sigma = \Gamma \times (0, T)$. Soit ω un ouvert non vide de Ω tel que $\overline{\omega} \subseteq \Omega$. On note par χ_ω la fonction caractéristique de ω. Nous considérons le système

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quasi-linéaire parabolique à contrôler suivant :

$$\begin{cases} y_t - \sum_{i,j=1}^n (a^{ij}(y)y_{x_i})_{x_j} = \chi_\omega u & \text{dans } Q, \\ y = 0 & \text{sur } \Sigma, \\ y(0) = y_0 & \text{dans } \Omega, \end{cases} \tag{1}$$

où $y = y(x,t)$ est l'état du système, $u = u(x,t)$ est la fonction contrôle, $a^{ij}(\cdot):\mathbb{R} \to \mathbb{R}$ sont des fonctions deux fois continûment dérivables vérifiant $a^{ij} = a^{ji}$ $(i, j = 1, \ldots, n)$, et pour une constante $\rho > 0$,

$$\sum_{i,j=1}^n a^{ij}(s)\xi_i\xi_j \geqslant \rho|\xi|^2, \quad \forall(s,\xi) = (s, \xi_1, \ldots, \xi_n) \in \mathbb{R} \times \mathbb{R}^n.$$

Le but de cette Note consiste à démontrer que le système (1) est localement contrôlable vers zéro, avec une estimation explicite du coût. Le résultat principal s'énonce comme suit :

Théorème 0.1. *Il existe une constante $\gamma > 0$ telle que, pour toute donnée initiale $y_0 \in C^{2+\frac{1}{2}}(\overline{\Omega})$ vérifiant $|y_0|_{C^{2+\frac{1}{2}}(\overline{\Omega})} \leqslant \gamma$ et la condition de compatibilité du premier ordre, il existe une fonction contrôle $u \in C^{\frac{1}{2},\frac{1}{4}}(\overline{Q})$ avec $\operatorname{supp} u \subseteq \omega \times [0,T]$ telle que la solution y du système (1) vérifie*

$$y(T) = 0 \quad \textit{dans } \Omega.$$

De plus,

$$|u|_{C^{\frac{1}{2},\frac{1}{4}}(\overline{Q})} \leqslant C_1 e^{e^{C_2 A}} |y_0|_{L^2(\Omega)},$$

où $A = \sum_{i,j=1}^n (1 + \sup_{|s|\leqslant 1} |a^{ij}(s)|^2 + \sup_{|s|\leqslant 1} |(a^{ij})'(s)|^2)$, $\nu = \max_{i,j=1,\ldots,n} \sup_{|s|\leqslant 1} |(a^{ij})'(s)|$, C_1 ne dépend que de ρ, ν, n, Ω et T, et C_2 ne dépend que de ρ, n, Ω et T.

1. Introduction and the main result

Let Ω be a bounded domain in $\mathbb{R}^n$ with C^3 boundary Γ. Put $Q = \Omega \times (0,T)$ and $\Sigma = \Gamma \times (0,T)$. Let ω be a nonempty open subset of Ω such that $\overline{\omega} \subseteq \Omega$. Denote by χ_ω the characteristic function of ω. We consider the following controlled quasilinear parabolic system:

$$\begin{cases} y_t - \sum_{i,j=1}^n (a^{ij}(y)y_{x_i})_{x_j} = \chi_\omega u & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y_0 & \text{in } \Omega, \end{cases} \tag{2}$$

where $y = y(x,t)$ is the *state variable*, $u = u(x,t)$ is the *control variable*, $a^{ij}(\cdot):\mathbb{R} \to \mathbb{R}$ are twice continuously differentiable functions satisfying $a^{ij} = a^{ji}$ $(i, j = 1, \ldots, n)$, and for some constant $\rho > 0$,

$$\sum_{i,j=1}^n a^{ij}(s)\xi_i\xi_j \geqslant \rho|\xi|^2, \quad \forall(s,\xi) = (s, \xi_1, \ldots, \xi_n) \in \mathbb{R} \times \mathbb{R}^n.$$

In what follows, we put

$$A = \sum_{i,j=1}^n \Big(1 + \sup_{|s|\leqslant 1} \big|a^{ij}(s)\big|^2 + \sup_{|s|\leqslant 1} \big|\big(a^{ij}\big)'(s)\big|^2\Big), \qquad \nu = \max_{i,j=1,\ldots,n} \sup_{|s|\leqslant 1} \big|\big(a^{ij}\big)'(s)\big|.$$

For any $k, l \in \mathbb{N}$, we denote by $C^{k,l}(\overline{Q})$ the set of all functions which have continuous derivatives up to order k with respect to the space variable and up to order l with respect to the time variable, and by $C^k(\overline{\Omega})$ the set of all functions which have continuous derivatives in $\overline{\Omega}$ up to order k. For any $\theta \in (0,1)$, put

$$C^{k+\theta,l+\frac{\theta}{2}}(\overline{Q}) = \left\{ f \in C^{k,l}(\overline{Q});\ \sup_{|\beta|=k} \sup_{(x,t)\neq(y,s)} \frac{|\partial_x^\beta \partial_t^l f(x,t) - \partial_x^\beta \partial_t^l f(y,s)|}{(|x-y| + |t-s|^{1/2})^\theta} < +\infty \right\},$$

and

$$C^{2+\theta}(\overline{\Omega}) = \left\{ f \in C^2(\overline{\Omega});\ \sup_{|\beta|=2} \sup_{x\neq y} \frac{|\partial_x^\beta f(x) - \partial_x^\beta f(y)|}{|x-y|^\theta} < +\infty \right\},$$

both of which are Banach spaces with canonical norms.

The purpose of this Note is to prove that system (2) is locally null controllable, with an explicit cost estimate on some null-control. Our main result is stated as follows.

Theorem 1.1. *There is a constant $\gamma > 0$ such that, for any initial value $y_0 \in C^{2+\frac{1}{2}}(\overline{\Omega})$ satisfying $|y_0|_{C^{2+\frac{1}{2}}(\overline{\Omega})} \leqslant \gamma$ and the first order compatibility condition, one can find a control $u \in C^{\frac{1}{2},\frac{1}{4}}(\overline{Q})$ with $\operatorname{supp} u \subseteq \omega \times [0,T]$ so that the solution y of system* (2) *satisfies*

$$y(T) = 0 \quad \text{in } \Omega.$$

Moreover,

$$|u|_{C^{\frac{1}{2},\frac{1}{4}}(\overline{Q})} \leqslant C_1 e^{e^{C_2 A}} |y_0|_{L^2(\Omega)}, \tag{3}$$

where C_1 depends only on ρ, ν, n, Ω and T, and C_2 depends only on ρ, n, Ω and T.

In the last decades, there are many papers devoted to the controllability of linear and semilinear parabolic equations (see e.g. [3–5,7,11] and the rich references therein). However, as far as we know, nothing is known about the controllability of quasilinear parabolic equations except for the case of one space dimension. In [2], by means of the fixed point method, the author proves the local null controllability of the diffusion equation (with homogeneous Dirichlet boundary condition) in one dimension, i.e., $y_t - (a(y))_{xx} = \chi_\omega u$ in Q and $y|_\Sigma = 0$, with $a(\cdot) \in C^3(\mathbb{R})$ satisfying $\inf_{s\in\mathbb{R}} a'(s) > 0$ and $\sup_{s\in\mathbb{R}}\{|a'(s)|, |a''(s)|, |a'''(s)|\} < +\infty$. In order to establish the desired estimates for weak solutions of the corresponding linearized equation, the author essentially makes use of the Sobolev embedding relation $L^\infty(0,T; H_0^1(\Omega)) \subseteq L^\infty(Q)$, which is valid only for one space dimension. Therefore, the same argument in [2] does not work in multidimensional space. To overcome this difficulty and establish a similar result for multidimensional case, we need to consider the problem in the frame of classical solutions. The key observation of our approach is that, for smooth initial data, the regularity of the null-control function for the linearized system can be improved, and therefore, the fixed point method is also applicable for multidimensional case.

We refer to [9] for the detailed proof of Theorem 1.1 and other related results.

2. Sketch of the proof of the main result

First, we derive a Carleman estimate for the following linear parabolic equation:

$$\begin{cases} p_t + \sum_{i,j=1}^n (b^{ij} p_{x_i})_{x_j} = 0 & \text{in } Q, \\ p = 0 & \text{on } \Sigma, \\ p(T) = p_T & \text{in } \Omega, \end{cases} \tag{4}$$

where $b^{ij} \in C^1(\overline{Q})$, $b^{ij} = b^{ji}$ $(i,j = 1,\dots,n)$, and for some constant $\rho > 0$,

$$\sum_{i,j=1}^n b^{ij}(x,t)\xi_i\xi_j \geqslant \rho|\xi|^2, \quad \forall (x,t,\xi) = (x,t,\xi_1,\dots,\xi_n) \in Q \times \mathbb{R}^n.$$

Put

$$B = 1 + \sum_{i,j=1}^n \left|b^{ij}\right|^2_{C^1(\overline{Q})}.$$

It is well known that [7], there exists a function $\psi \in C^2(\overline{\Omega})$ such that

$$\psi(x) > 0, \quad \text{in } \Omega; \qquad \psi(x) = 0, \quad \text{on } \partial\Omega; \qquad \left|\nabla\psi(x)\right| > 0, \quad \text{in } \overline{\Omega} \setminus \omega_0,$$

where ω_0 is any given open and nonempty subset of ω such that $\overline{\omega_0} \subseteq \omega$. For any given parameter $\mu > 0$, put

$$\varphi \equiv \varphi(x,t) = \frac{e^{\mu\psi(x)}}{t(T-t)}, \qquad \alpha \equiv \alpha(x,t) = \frac{e^{\mu\psi(x)} - e^{2\mu|\psi|_{C(\overline{\Omega})}}}{t(T-t)}.$$

Let ω_1 be any fixed open subset of ω such that $\overline{\omega_0} \subseteq \omega_1$ and $\overline{\omega_1} \subseteq \omega$. We have the following global Carleman estimate for Eq. (4):

Proposition 2.1. *There exist two constants $C > 0$ and $C_0 > 0$, depending only on ρ, n, Ω and T, such that for any $\mu \geqslant C_0 B$ and $\lambda \geqslant C_0 e^{2\mu|\psi|_{C(\overline{\Omega})}}$, solutions of Eq.* (4) *satisfy*

$$\int_Q \left\{ e^{2\lambda\alpha} \frac{1}{\lambda\varphi} \left[p_t^2 + \left| \sum_{i,j=1}^n \left(b^{ij} p_{x_i}\right)_{x_j} \right|^2 \right] + e^{2\lambda\alpha} \lambda\mu^2\varphi|\nabla p|^2 + e^{2\lambda\alpha}\lambda^3\mu^4\varphi^3 p^2 \right\} dx\,dt$$
$$\leqslant C \left(1 + \sum_{i,j=1}^n \left|b^{ij}\right|^2_{C(\overline{Q})} + \sum_{i,j=1}^n \left|b_t^{ij}\right|_{C(\overline{Q})} \right)^2 \int_0^T \int_{\omega_1} e^{2\lambda\alpha}\lambda^3\mu^4\varphi^3 p^2\,dx\,dt, \quad \forall p_T \in L^2(\Omega).$$

The proof of Proposition 2.1 is based on a new pointwise estimate for the parabolic operator, which can be found in [9]. This pointwise estimate is a little different from those in [6] and [10], such that the constants appeared in the above Carleman inequality depend only on $|b^{ij}|_{C^1(\overline{Q})}$ instead of $|b^{ij}|_{C^{2,1}(\overline{Q})}$.

By Proposition 2.1 and applying the usual energy estimate to Eq. (4), it is easy to show the following observability inequality:

Proposition 2.2. *There exist two constants $C > 0$ and $C_0 > 0$, depending only on ρ, n, Ω and T, such that the following estimate*

$$\int_\Omega p^2(0)\,dx \leqslant C e^{e^{CB}} \int_0^T \int_{\omega_1} e^{2\lambda\alpha}\varphi^3 p^2\,dx\,dt$$

holds for all solutions of (4) *and $\lambda \geqslant C_0 e^{C_0 B}$.*

Remark 1. The novelty in the above observability inequality is the explicit estimate on the observability constant $Ce^{e^{CB}}$ in terms of the C^1-norms of the coefficients in the principle operator appeared in the first equation of (4).

Next, we establish the null controllability of the linearized system of (2). For this, for any fixed $z \in C^{1+\frac{1}{2},1+\frac{1}{4}}(\overline{Q})$ with $z(0) = y_0$ in Ω, consider the following linear controlled system

$$\begin{cases} y_t - \sum_{i,j=1}^n (a^{ij}(z) y_{x_i})_{x_j} = \xi u & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y_0 & \text{in } \Omega, \end{cases} \tag{5}$$

where $\xi \in C_0^\infty(\omega)$ and $\xi = 1$ in ω_1, and as before y is the *state variable*, u is the *control variable*.

Using Proposition 2.2 and adopting the iteration method used in [1], we have the following null controllability result for system (5) with "smooth" control function:

Proposition 2.3. *For any $y_0 \in C^{2+\frac{1}{2}}(\overline{\Omega})$ satisfying the first order compatibility condition, there exists a control $u \in C^{\frac{1}{2},\frac{1}{4}}(\overline{Q})$ such that the corresponding solution y of system* (5) *satisfies*

$$y(T) = 0 \quad \text{in } \Omega.$$

Moreover,

$$|u|_{C^{\frac{1}{2},\frac{1}{4}}(\overline{Q})} \leqslant C_1 \exp\left\{ \exp\left\{ C_2 \sum_{i,j=1}^n \left(1 + \left|a^{ij}(z)\right|^2_{C^1(\overline{Q})}\right) \right\} \right\} |y_0|_{L^2(\Omega)}, \tag{6}$$

where C_1 is a constant depending only on ρ, $\max_{i,j=1,\dots,n}\sup_Q|\nabla_{x,t}a^{ij}(z)|$, n, Ω and T, and C_2 is a constant depending only on ρ, n, Ω and T.

In what follows, we give a proof of the main result in this Note.

Proof of Theorem 1.1. First, set

$$K=\left\{z\in C^{1+\frac{1}{2},1+\frac{1}{4}}(\overline{Q});\ |z|_{C^{1+\frac{1}{2},1+\frac{1}{4}}(\overline{Q})}\leqslant 1,\ z(0)=y_0\right\}.$$

Clearly, K is a nonempty convex and compact subset of $L^2(Q)$ for small initial datum y_0. For $z\in K$, put

$$\Phi(z)=\left\{y\in K;\ \exists u\in C^{\frac{1}{2},\frac{1}{4}}(\overline{Q})\text{ and a constant }C\text{ such that }(u,y)\text{ satisfies (5), (6) and }y(T)=0\text{ in }\Omega\right\}.$$

This define a (possibly multi-valued) map $\Phi: K\to 2^K$, provided that y_0 is small enough in $C^{2+\frac{1}{2}}(\overline{\Omega})$. Indeed, by Proposition 2.3, there exists a control $u\in C^{\frac{1}{2},\frac{1}{4}}(\overline{Q})$ satisfying (6), such that the solution y of system (5) satisfies $y(T)=0$ in Ω, and therefore, by the Schauder theory for linear parabolic equations (see [8]), it follows

$$|y|_{C^{2+\frac{1}{2},1+\frac{1}{4}}(\overline{Q})}\leqslant C_3\left(|y_0|_{C^{2+\frac{1}{2}}(\overline{\Omega})}+|\xi u|_{C^{\frac{1}{2},\frac{1}{4}}(\overline{Q})}\right),$$

where C_3 depends on n,Ω,T and $|a^{ij}(z)|_{C^{1+\frac{1}{2},\frac{1}{4}}(\overline{Q})}$. Since $z\in K$, it follows

$$|y|_{C^{2+\frac{1}{2},1+\frac{1}{4}}(\overline{Q})}\leqslant C_4\left(|y_0|_{C^{2+\frac{1}{2}}(\overline{\Omega})}+e^{e^{C_2A}}|y_0|_{L^2(\Omega)}\right),$$

where C_4 depends on $\sum_{i,j=1}^n\sup_{|s|\leqslant 1}(|a^{ij}(s)|^2+|(a^{ij})'(s)|^2+|(a^{ij})''(s)|^2)$. Consequently, there exists a sufficiently small constant $\gamma>0$, such that $|y|_{C^{2+\frac{1}{2},1+\frac{1}{4}}(\overline{Q})}\leqslant 1$ whenever $|y_0|_{C^{2+\frac{1}{2}}(\overline{\Omega})}\leqslant\gamma$, i.e., $y\in K$. Hence Φ is well defined for small initial datum y_0.

Further, for any $z\in K$, $\Phi(z)$ is a nonempty convex and compact subset of $L^2(Q)$. Also, Φ is lower semicontinuous. Therefore, by Kakutani's fixed point theorem, there exists a $y\in K$, such that $y\in\Phi(y)$. This means that for system (2), there is a control $u\in C^{\frac{1}{2},\frac{1}{4}}(\overline{\Omega})$ with $\operatorname{supp}u\subseteq\omega\times(0,T)$, such that the solution of (2) satisfies $y(T)=0$ in Ω. On the other hand, by Proposition 2.3, the cost of the control function verifies (3). This completes the proof of Theorem 1.1. □

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