Partial Differential Equations

Alexandroff–Bakelman–Pucci estimate for singular or degenerate fully nonlinear elliptic equations

Gonzalo Dávila [a,1], Patricio Felmer [a,2], Alexander Quaas [b,3]

[a] *Departamento de Ingeniería Matemática and Centro de Modelamiento Matemático, UMR2071 CNRS-UChile, Universidad de Chile, Santiago, Chile*

[b] *Departamento de Matemática, Universidad Técnica Federico Santa María, Av. Espana 1680, V-110 Valparaiso, Chile*

Abstract

We prove the classical Alexandroff–Bakelman–Pucci estimate for fully nonlinear elliptic equations involving singular or degenerate operators having as models $|p|^\alpha \mathcal{M}_{\lambda,\Lambda}^{\pm}(X)$, where $\mathcal{M}_{\lambda,\Lambda}^{\pm}$ are the Pucci extremal operators with parameters $0 < \lambda \leqslant \Lambda$ and $\alpha > -1$. ***To cite this article: G. Dávila et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***
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Résumé

Inégalité d'Alexandroff–Bakelman–Pucci pour des équations elliptiques entièrement non linéaires singulières ou dégénérés. Nous prouvons l'inégalité classique d'Alexandroff–Bakelman–Pucci pour des équations elliptiques entièrement non linéaires avec des opérateurs singulières ou dégénérés ayant comme modèles $|p|^\alpha \mathcal{M}_{\lambda,\Lambda}^{\pm}(X)$ où $\mathcal{M}_{\lambda,\Lambda}^{\pm}$ sont les opérateurs extremal de Pucci avec des paramètres $0 < \lambda \leqslant \Lambda$ et $\alpha > -1$. ***Pour citer cet article : G. Dávila et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***
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In this Note we establish the classical ABP estimate for fully nonlinear elliptic equations of the form

$$F(\nabla u, D^2 u) + b \cdot \nabla u |\nabla(u)|^\alpha + cu|u|^\alpha = f \quad \text{in } \Omega, \tag{1}$$

where $\alpha > -1$, $\Omega \subseteq \mathbb{R}^n$ is a bounded domain, $b, c : \overline{\Omega} \to \mathbb{R}^n$ and $f : \Omega \to \mathbb{R}$ are continuous functions. Our model for F is the operator $F(p, X) = |p|^\alpha \mathcal{M}_{\lambda,\Lambda}^{\pm}(X)$, where $\mathcal{M}_{\lambda,\Lambda}^{\pm}$ are the Pucci operators. The mean curvature and the p-Laplacian operators also satisfy our hypothesis. More generally $F : (\mathbb{R}^n \setminus \{0\}) \times \mathcal{S}(n) \to \mathbb{R}$ is a continuous function and it satisfies:

(H1) $F(tp, \mu X) = |t|^\alpha \mu F(p, X)$, $\forall t \in \mathbb{R} \setminus \{0\}$, $\mu \in \mathbb{R}^+$.

(H2) There exists $\Lambda \geqslant \lambda > 0$ such that $\forall p \neq 0, \forall (M, N) \in \mathcal{S}^2(n)$

$$|p|^\alpha \mathcal{M}_{\lambda,\Lambda}^-(N) \leqslant F(p, M + N) - F(p, M) \leqslant |p|^\alpha \mathcal{M}_{\lambda,\Lambda}^+(N).$$

We also assume that $|c(x)|, |b(x)| \leqslant \gamma, \forall x \in \overline{\Omega}, \gamma > 0$. Here $\mathcal{S}(n)$ is the set of symmetric $n \times n$ matrices.

Equations in the form of (1) have been studied recently by Birindelli and Demengel in a series of papers [3–7], where existence results, principal eigenvalues, comparison for the Dirichlet and other interesting properties are established. In these papers the notion of viscosity solution considered is that of Chen, Giga and Goto [10] and Evans and Spruck [12], a slight variation of the usual one, that only tests when the gradient of the test function does not vanish. In this Note we use this notion.

The aim of this Note is to prove a basic qualitative property known as ABP estimate for (1). For the linear case see [13] and for the fully nonlinear uniform elliptic case ($\alpha = 0$), the result can be found in [8] and [9]. Precisely we prove:

Theorem 1. *Under the general conditions given above, in particular $\alpha > -1$ and F satisfying* (H1) *and* (H2)*, and additionally with $c \leqslant 0$, there exists $C = C(n, \lambda, \alpha, \gamma, \mathrm{diam}(\Omega))$ such that for any $u \in C(\overline{\Omega})$ viscosity sub-solution* (*resp. super-solution*) *of* (1) *in $\{x \in \Omega \mid 0 < u(x)\}$ (resp. $\{x \in \Omega \mid 0 > u(x)\}$), satisfies*

$$\sup_\Omega u \leqslant \sup_{\partial\Omega} u^+ + C \cdot \mathrm{diam}(\Omega) \| f^- \|_{L^n(\Gamma^+(u^+))}^{\frac{1}{1+\alpha}} \quad \left(resp. \ \sup_\Omega u^- \leqslant \sup_{\partial\Omega} u^- + C \cdot \mathrm{diam}(\Omega) \| f^+ \|_{L^n(\Gamma^+(u^-))}^{\frac{1}{1+\alpha}} \right).$$

Here $\Gamma^+(u) = \{x \in \Omega \colon \exists p \text{ such that } u(y) \leqslant u(x) + \langle p, y - x \rangle \ \forall y \in \Omega\}$.

Given a real number x we write $x^\pm = \max\{0, \pm x\}$. Here we are assuming that the coefficients and the right-hand side of the equation are continuous functions, but in view of the work by Crandall, Caffarelli, Kocan and Swiech [9], one could expect the result is also true for measurable ingredients. However at this point this cannot be done, since there is no L^p notion of solution for this kind of equations.

We notice that ABP inequality implies the maximum principle for small domains, without the sign restriction on c, as first observed by Bakelman and extensively used in [2] and many others. It will be natural to ask if ABP estimate holds for equations having positive eigenvalue (see [1] for the linear case and [16] for the fully nonlinear case with $\alpha = 0$), in that direction we have the following partial result:

Corollary 2. *Under the hypotheses of Theorem* 1*, but without sign constraint on c, there exist $C = C(n, \lambda, \alpha, \gamma, \mathrm{diam}(\Omega))$ and $\varepsilon > 0$ such that for any $u \in C(\overline{\Omega})$ viscosity sub-solution of* (1)*, with $c \leqslant \varepsilon$, we have*

$$\sup_\Omega u \leqslant C \cdot \left(\sup_{\partial\Omega} u^+ + \mathrm{diam}(\Omega) \| f^- \|_{L^n(\Gamma^+(u^+))}^{\frac{1}{1+\alpha}} \right).$$

The proof of Theorem 1 is based on two regularization procedures (sup-convolution and standard mollification) as in [9]. This approximation procedure only holds for continuous function F, so in the case $-1 < \alpha < 0$ we need some extra work, see Lemma 4. We would like to mention that after submitting this Note we have learned of papers by Junges Miotto [15] and Imbert [14], where similar results are obtained.

Before presenting a proof of Theorem 1, it is convenient to recall the definition of solution for (1).

Definition 3. We say that $u \in C(\Omega)$ is a viscosity sub- (super-)solution of (1) in Ω if for all $x_0 \in \Omega$:

(i) Either for all $\varphi \in C^2(\Omega)$ such that $u - \varphi$ has a local maximum (minimum) on x_0 and $\nabla\varphi(x_0) \neq 0$ one has

$$F\big(\nabla\varphi(x_0), D^2\varphi(x_0)\big) + b(x_0) \cdot \nabla\varphi(x_0) \big| \nabla\varphi(x_0) \big|^\alpha + c(x_0) u(x_0) \big| u(x_0) \big|^\alpha \geqslant f(x_0) \quad (\text{resp. } \leqslant).$$

(ii) Or there exists a open ball $B(x_0, \delta) \subset \Omega$, $\delta > 0$, such that $u = C$ in $B(x_0, \delta)$ and

$$c(x) C |C|^\alpha \geqslant f(x) \quad \forall x \in B(x_0, \delta) \quad (\text{resp. } \leqslant).$$

Remark 1. When F is continuous, this definition is equivalent to the usual one as it is proved in [11].

Since our approximation procedure only applies to continuous operators, we need the following lemma:

Lemma 4. *Let* $-1 < \alpha < 0$, $f \leqslant 0$ *and* u *be a sub-solution of*

$$|\nabla u|^{\alpha} \mathcal{M}^{+}(D^2 u) + \gamma |\nabla u|^{\alpha+1} + \gamma |u|^{\alpha+1} = f,$$

in the sense of Definition 3, *then for all* $\sigma > 0$, u *is a sub-solution of*

$$\big(|\nabla u| + \sigma\big)^{\alpha} \mathcal{M}^{+}(D^2 u) + \gamma |\nabla u|^{\alpha+1} + \gamma |u|^{\alpha+1} = f$$

in the usual sense (*from now on we omit the parameters from the Pucci operators*).

Proof. In case u is not constant let $\varphi \in C^2(\Omega)$ a test function at $x_0 \in \Omega$. By Remark 1 we may assume that $\nabla\varphi(x_0) \neq 0$, so that in case $\mathcal{M}^{+}(D^2\varphi(x_0)) \leqslant 0$, we obtain

$$\big(\big|\nabla\varphi(x_0)\big| + \sigma\big)^{\alpha} \mathcal{M}^{+}\big(D^2\varphi(x_0)\big) + \gamma \big|\nabla\varphi(x_0)\big|^{\alpha+1} + \gamma \big|\varphi(x_0)\big|^{\alpha+1} \geqslant f(x_0)$$

since $\alpha \in (-1, 0)$ and u is a viscosity solution is the sense of Definition 3. If $\mathcal{M}^{+}(D^2\varphi(x_0)) > 0$, then we get the inequality using that $f \leqslant 0$. The result in case u constant is trivial. $\square$

Proof of Theorem 1. We only prove one inequality since $\mathcal{M}^{+}(-X) = -\mathcal{M}^{-}(X)$, for all $X \in \mathcal{S}(n)$.

Case $\alpha > 0$: First we assume that $u \in C^2(\Omega) \cap C(\overline{\Omega})$, so that the inequality in (1) is satisfied pointwise. For $r < r_0 := (\sup_{\Omega} u - \sup_{\partial\Omega} u^{+})/(\operatorname{diam}(\Omega))$ we define the set

$$\Gamma_r^{+}(u) = \big\{x \in \Omega \colon \exists p \in B_r(0) \text{ such that } u(y) \leqslant u(x) + \langle p, y - x \rangle \, \forall y \in \Omega\big\}.$$

It is not hard to see that $\Gamma_r^{+}(u)$ is a compact subset of Ω, that $B_r(0) = Du(\Gamma_r^{+}(v))$ and that $D^2 u(x) \leqslant 0$ in $\Gamma_r^{+}(u) \subset \{0 < u\}$. Hence for $\kappa \geqslant 0$ the change of variable $p = \nabla u(x)$ implies

$$I := \int_{B_r} \big(|p|^{\frac{n}{n-1}} + \kappa^{\frac{n}{n-1}} |p|^{\frac{-\alpha n}{n-1}}\big)^{1-n} \, \mathrm{d}p \leqslant \int_{\Gamma_r^{+}(u)} \big(|\nabla u|^{\frac{n}{n-1}} + \kappa^{\frac{n}{n-1}} |\nabla u|^{\frac{-\alpha n}{n-1}}\big)^{1-n} \left(-\frac{\operatorname{tr}(D^2 u)}{n}\right)^{n} \mathrm{d}x. \tag{2}$$

We observe that the function under the integral sign on the right vanishes when the gradient of u vanishes. Therefore, at points where the gradient of u does not vanish, we can multiply the inequality in (1) by $|\nabla u|^{-\alpha}$ and then use that $D^2 u \leqslant 0$ in Γ_r^{+}, $b(x) \leqslant \gamma$ and Hölder inequality to obtain

$$I \leqslant \frac{1}{n^n \lambda^n} \int_{\Gamma_r^{+}(u)} \big(|\nabla u|^{\frac{n}{n-1}} + \kappa^{\frac{n}{n-1}} |\nabla u|^{\frac{-\alpha n}{n-1}}\big)^{1-n} \big(\gamma |\nabla u| + f^{-} |\nabla u|^{-\alpha}\big)^{n} \, \mathrm{d}x$$

$$\leqslant \frac{1}{n^n \lambda^n} \int_{\Gamma_r^{+}(u)} \left(\gamma^n + \frac{(f^{-}(x))^n}{\kappa^n}\right) \mathrm{d}x.$$

On the other hand, by a direct computation and using the inequality $(a+b)^k \leqslant 2^{k-1}(a^k + b^k)$ which is true for $a, b \geqslant 0$ and $k \in \mathbb{N}$, we obtain

$$\frac{2^{2-n}}{n(\alpha+1)} \omega_n \ln\left(\frac{r^{n(\alpha+1)}}{\kappa^n} + 1\right) = 2^{2-n} \int_{B_r} \frac{1}{|p|^n + \kappa^n |p|^{-\alpha n}} \, \mathrm{d}p \leqslant \int_{B_r} \big(|p|^{\frac{n}{n-1}} + \kappa^{\frac{n}{n-1}} |p|^{\frac{-\alpha n}{n-1}}\big)^{1-n} \, \mathrm{d}p.$$

Then we choose $\kappa = \|f^{-}\|_{L^n(\Gamma_r^{+}(u))}/\lambda \neq 0$ and we combine the last two inequalities to get $r \leqslant C\|f^{-}\|^{\frac{1}{\alpha+1}}$, from where the inequality follows. If $u \in C(\overline{\Omega})$ we use the approximation procedure devised in [9], defining the sup-convolution $u_{\varepsilon}(x) = \sup_{y \in B}(u(y) - |x - y|^2/\varepsilon)$ of the function u, whose properties are given in detail in [9]. The standard mollification u_{ε}^{ρ} satisfies (2) with the obvious changes and then u_{ε} also satisfies it. Here we use that $\Gamma_r^{+}(u)$ and $\Gamma_r^{+}(u_{\varepsilon})$ belong to a common compact set, as proved in [9]. Since u_{ε} satisfies the inequality in (1) in an approximate fashion (see Lemma A.3 in [9]) from (2) it follows the result, after taking limit as $\varepsilon \to 0$.

Case $\alpha \in (-1, 0)$: We consider $\sigma > 0$ and we use Lemma 4 to get a continuous operator in our equation for u. Then we can do the same proof as in the previous case, but with $|p| + \sigma$ instead of $|p|$. Finally we take limit as $\sigma \to 0$ and we conclude. $\square$

Proof of Corollary 2. In Eq. (1) we consider $c = 0$ and the new right-hand side $\tilde{f} = f - \varepsilon |u|^{\alpha+1} \leqslant f - c(x)u|u|^{\alpha}$, where ε is given in the hypothesis. Then we apply Theorem 1 to this problem and get $C \cdot \mathrm{diam}(\Omega)(\|f^-\|_{L^n(\Gamma^+(u^+))}^{\frac{1}{1+\alpha}} + \varepsilon^{\frac{1}{1+\alpha}} \sup_\Omega u)$ on the right-hand side. From here we see that if $\varepsilon > 0$ is so that $C \cdot \mathrm{diam}(\Omega)\varepsilon^{\frac{1}{1+\alpha}} < 1$, then the required constant can be chosen. $\square$

References

[1] H. Berestycki, L. Nirenberg, S.R.S. Varadhan, The principal eigenvalue and maximum principle for second-order elliptic operators in general domains, Comm. Pure Appl. Math. 47 (1) (1994) 47–92.
[2] H. Berestycki, L. Nirenberg, On the method of moving planes and the sliding method, Boll. Soc. Brasil Mat. (N.S.) 22 (1991) 237–275.
[3] I. Birindelli, F. Demengel, Comparison principle and Liouville type results for singular fully nonlinear operators, Ann. Fac. Sci Toulouse Math. (6) 13 (2) (2004) 261–287.
[4] I. Birindelli, F. Demengel, Eigenvalue and maximum principle for fully nonlinear singular operators, Adv. Partial Differ. Equ. 11 (1) (2006) 91–119.
[5] I. Birindelli, F. Demengel, Eigenvalue, maximum principle and regularity for fully nonlinear homogeneous operators, Commun. Pure Appl. Anal. 6 (2007) 335–366.
[6] I. Birindelli, F. Demengel, The Dirichlet problem for singular fully nonlinear operators, Discrete Contin. Dyn. Syst. (special vol.) (2007) 110–121.
[7] I. Birindelli, F. Demengel, Eigenvalue and Dirichlet problem for fully-nonlinear operators in non-smooth domains, J. Math. Anal. Appl. 352 (2) (2009) 822–835.
[8] L. Caffarelli, X. Cabré, Fully Nonlinear Elliptic Equations, Colloquium Publications, vol. 43, American Mathematical Society, 1995.
[9] L. Caffarelli, M. Crandall, M. Kocan, A. Świech, On viscosity solutions of fully nonlinear equations with measurable ingredients, Comm. Pure Appl. Math. 49 (4) (1996) 365–398.
[10] Y.G. Chen, Y. Giga, S. Goto, Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations, J. Differential Geom. 33 (1991) 749–786.
[11] G. Dávila, P. Felmer, A. Quaas, Harnack inequality for singular fully nonlinear operators and some existence results, preprint.
[12] C. Evans, J. Spruck, Motion of level sets by mean curvature, J. Differential Geom. 33 (1991) 635–681.
[13] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, 2nd edition, Springer-Verlag, 1983.
[14] C. Imbert, Alexandroff–Bakelman–Pucci estimate and Harnack inequality for degenerate fully non-linear elliptic equations, preprint.
[15] T. Junges Miotto, The Aleksandrov–Bakelman–Pucci estimate for singular fully nonlinear operators, preprint.
[16] A. Quaas, B. Sirakov, On the principal eigenvalues and the Dirichlet problem for fully nonlinear operators, Adv. Math. 218 (1) (2008) 105–135.