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C. R. Acad. Sci. Paris, Ser. I 347 (2009) 1141–1146
COMPTES RENDUS MATHEMATIQUE

Partial Differential Equations/Mathematical Problems in Mechanics

On viscosity-continuous solutions of the Euler and Navier–Stokes equations with a Navier-type boundary condition

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Received 6 June 2009; accepted after revision 9 September 2009

Available online 30 September 2009

Presented by Gérard Iooss

Abstract

Provided the initial velocity and the external body force are sufficiently smooth, there exist $T_0 > 0$, $\nu^* > 0$ and a unique continuous family of strong solutions $\mathbf{u}^\nu$ $(0 \leqslant \nu < \nu^*)$ of the Euler or Navier–Stokes initial–boundary value problem on the time interval $(0, T_0)$. The solutions of the Navier–Stokes problem satisfy a Navier-type boundary condition. ***To cite this article: H. Bellout et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Solutions continues en la viscosité pour les équations d'Euler ou de Navier–Stokes avec des conditions aux limites de type Navier. Pourvu que les données soient suffisamment régulières, il existe $T_0 > 0$, $\nu^* > 0$ et $\{u^\nu\}_{0 \leqslant \nu < \nu^*}$ famille unique de solutions fortes, locales en temps sur $(0, T_0)$ et dépendant continûment de ν, pour les problèmes d'Euler ou de Navier–Stokes. Ces solutions vérifient des conditions aux limites de type celles de Navier. ***Pour citer cet article : H. Bellout et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Version française abrégée

Soit $Q_T = \Omega \times (0, T)$ où $T > 0$ et Ω un domaine borné de $\mathbb{R}^3$ de frontière $\partial\Omega$ de classe $C^{3,1}$. Dans Q_T, l'écoulement d'un fluide Newtonien visqueux incompressible est usuellement décrit par les équations de Navier–Stokes

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \nu \Delta \mathbf{u} + \mathbf{f}, \qquad \operatorname{div} \mathbf{u} = 0.$$

Les équations d'Euler correspondent formellement au cas inviscide, disons au cas limite où le coefficient de viscosité ν tend vers 0. Nous leur associons la donnée initiale $\mathbf{u}(\cdot, 0) = \mathbf{u}^*$ et nous supposons la condition aux limites de flux nul $\mathbf{u} \cdot \mathbf{n} = 0$ sur $\partial\Omega \times (0, T)$ pour nous placer dans le cadre des résultats classiques d'existence [6,10] : « *il existe*

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doi:10.1016/j.crma.2009.09.007

$T_0 \in (0,T)$ et il existe $\mathbf{u}^0$ unique solution du modèle d'Euler dans $L^\infty(0,T_0;\mathbf{W}^{4,2}(\Omega))$ pourvu que les données soient $\mathbf{f} \in L^1(0,T;\mathbf{W}^{4,2}(\Omega))$ et $\mathbf{u}^* \in \mathbf{W}^{4,2}(\Omega) \cap \mathbf{L}^2_\sigma(\Omega)$ tel que $\mathbf{u}^* \cdot \mathbf{n} = 0$ sur $\partial\Omega$ et $\operatorname{div}\mathbf{u}^* = 0$ dans Ω. p^0 est le champ de pression associé».

Le problème que nous considérons dans cette Note est :

Problème (P). *Trouver une condition aux limites (*) pour $\mathbf{u}|_{\partial\Omega}$, complémentaire de la condition de flux nul, permettant de prolonger $\mathbf{u}^0$ par une famille $\{\mathbf{u}^\nu\}$ de solutions fortes du modèle correspondant de Navier–Stokes, l'unicité et sur un certain intervalle $[0,\nu^*)$ la ν-continuité étant requises.*

Nous avons déjà résolu le problème (P) lorsque la condition (*) caractérise les flux de vorticité, i.e. $\mathbf{curl}^k\,\mathbf{u} \cdot \mathbf{n} = \mathbf{curl}^k(\mathbf{u}^0 + \nu\mathbf{v}^0) \cdot \mathbf{n}$ pour $k = 1, 2$ et $\mathbf{v}^0$ adéquat, voir [3] (et [4] pour l'analyse du modèle de Navier–Stokes avec de telles conditions aux limites).

Nous proposons maintenant de résoudre (P) lorsque la condition (*) est du type condition de Navier, i.e. $\mathbf{curl}\,\mathbf{u} \times \mathbf{n} = \mathbf{a}^\nu$, avec $\mathbf{a}^\nu := \mathbf{curl}(\mathbf{u}^0 + \nu\mathbf{v}^0) \times \mathbf{n}$. Comme dans [3], le point-clé est l'introduction de $(\mathbf{v}^0, p^0_v)$ afin de justifier la structure suivante de $(\mathbf{u}^\nu, p^\nu)$ solution unique du modèle de Navier–Stokes

$$\mathbf{u}^\nu = \mathbf{u}^0 + \nu\big(\mathbf{v}^0 + \mathbf{w}^\nu\big), \qquad p^\nu = p^0 + \nu\big(p^0_v + p^\nu_w\big),$$

où $(\mathbf{v}^0, p^0_v)$ résout le même problème qu'en [3], linéaire et de nature hyperbolique, cf. (8)–(10) dans la partie anglaise ; alors $(\mathbf{w}^\nu, p^\nu_w)$ résout un problème non linéaire qu'il est facile d'écrire.

La structure que nous choisissons pour $(\mathbf{u}^\nu, p^\nu)$ semble très naturelle, elle permet l'obtention de bonnes estimations et par voie de conséquence le résultat de convergence escompté dans une topologie forte appropriée lorsque $\nu \to 0$. Autre avantage, la technique d'analyse préconisée nous conduit, via $\mathbf{v}^0$ et $\mathbf{w}^\nu$, à démontrer effectivement l'existence de $\mathbf{u}^\nu$ sur un intervalle $(0, T(\nu))$ tel que $T(\nu) = T_0$. Le modèle linéaire choisi décrivant $\mathbf{v}^0$ est fondamental, son pilotage par $\Delta\mathbf{u}^0$ au second membre également (avec $\nu\mathbf{v}^0$ fonction de $\nu\Delta\mathbf{u}^0$, $\mathbf{u}^0$ étant solution du modèle d'Euler).

Les principaux résultats sont, le troisième apportant la réponse au problème (P) :

Théorème 1. *Il existe une solution unique $\mathbf{v}^0$ au problème (8)–(10), $\mathbf{v}^0 \in L^\infty(0,T_0;\mathbf{W}^{2,2}_\sigma(\Omega))$, le contrôle de sa norme $|||\mathbf{v}^0|||_{\infty;2,2}$ étant assuré en fonction de Ω, T_0 et $|||\mathbf{u}^0|||_{1;4,2}$.*

Théorème 2. *Il existe $\nu^* > 0$ et pour chaque $0 < \nu < \nu^*$, il existe $\mathbf{w}^\nu$ solution unique au problème non linéaire non explicité (cf. (11)–(13) dans la partie anglaise), $\mathbf{w}^\nu \in L^\infty(0,T_0\,;\mathbf{W}^{1,2}_\sigma(\Omega)) \cap L^2(0,T_0\,;\mathbf{W}^{2,2}(\Omega))$. Cette famille de solutions $\mathbf{w}^\nu$ dépend continûment de ν dans la norme $|||.|||_{\infty;1,2} + |||.|||_{2;2,2}$ et vérifie les estimations suivantes*

$$\big|\big|\big|\mathbf{w}^\nu\big|\big|\big|_{\infty;0,2} \leqslant c_1\nu, \qquad \big|\big|\big|\mathbf{w}^\nu\big|\big|\big|_{\infty;1,2} \leqslant c_2\sqrt{\nu}, \qquad \big|\big|\big|\mathbf{w}^\nu\big|\big|\big|_{2;2,2} \leqslant c_3,$$

les constantes c_1, c_2, c_3 ne dépendant pas de ν.

Théorème 3. *Les données sont $\mathbf{u}^* \in \mathbf{W}^{4,2}(\Omega) \cap \mathbf{L}^2_\sigma(\Omega)$ et $\mathbf{f} \in L^1(0,T_0;\mathbf{W}^{4,2}(\Omega))$. Alors il existe $T_0 \in (0,T)$ et $\nu^* > 0$ et $\{\mathbf{u}^\nu\}_{0\leqslant\nu<\nu^*}$ famille unique de solutions des problèmes d'Euler ou de Navier–Stokes, $\mathbf{u}^\nu \in L^\infty(0,T_0;\mathbf{W}^{1,2}_\sigma(\Omega)) \cap L^2(0,T_0;\mathbf{W}^{2,2}(\Omega))$, $\mathbf{u}^\nu$ dépendant continûment de ν dans la norme $|||.|||_{\infty;1,2} + |||.|||_{2;2,2}$, l'erreur $\mathbf{u}^\nu - \mathbf{u}^0$ étant de la forme $\nu\mathbf{v}^0 + \nu\mathbf{w}^\nu$ où $\mathbf{v}^0$ et $\mathbf{w}^\nu$ ont les propriétés énoncées aux théorèmes précédents.*

On trouvera dans la partie anglaise les grandes lignes des démonstrations (et tous les détails dans [5]). Cette Note se terminera par un commentaire sur l'influence du choix des conditions aux limites pour résoudre le problème (P), et une discussion de l' article [11] publié par Y.L. Xiao et Z.P. Xin.

1. Formulation of the problem

Notation. We denote vector functions and spaces of vector functions by boldface letters. Concretely, $\mathbf{L}^r(\Omega)$ (respectively $\mathbf{W}^{k,r}(\Omega)$) is the Lebesgue space (respectively the Sobolev space) of vector functions. Furthermore, we use the notation:

- $\mathbf{L}^2_\sigma(\Omega) :=$ the closure of $\{\mathbf{u} \in \mathbf{C}^\infty_0(\Omega);\ \operatorname{div}\mathbf{u} = 0 \text{ in } \Omega\}$ in $\mathbf{L}^2(\Omega)$. The norm is denoted by $\|.\|_2$.

- $\mathbf{W}^{k,2}_\sigma(\Omega) := \mathbf{W}^{k,2}(\Omega) \cap \mathbf{L}^2_\sigma(\Omega)$ for $k = 1, 2$. The norm is denoted by $\|.\|_{k,2}$.
- $\|\!|.\|\!|_{s;k,r}$ is the norm in $L^s(0,T;\mathbf{L}^r(\Omega))$ (if $k = 0$) or in $L^s(0,T;\mathbf{W}^{k,r}(\Omega))$ (if $k > 0$).

The Euler problem. *Suppose that $T > 0$ and Ω is a bounded domain in $\mathbb{R}^3$ with the boundary $\partial\Omega$ of the class $C^{3,1}$. Put $Q_T = \Omega \times (0,T)$. We recall the Euler initial–boundary value problem*

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = \mathbf{f}, \qquad \operatorname{div}\mathbf{u} = 0 \quad \text{in } Q_T, \tag{1}$$

$$\mathbf{u}(.,0) = \mathbf{u}^* \quad \text{in } \Omega, \tag{2}$$

$$\mathbf{u}\cdot\mathbf{n} = 0 \quad \text{on } \partial\Omega \times (0,T). \tag{3}$$

The general classical result of J.P. Bourguignon and H. Brezis [6] *and R. Temam* [10] *particularly says: If $\mathbf{u}^* \in \mathbf{W}^{4,2}(\Omega) \cap \mathbf{L}^2_\sigma(\Omega)$ satisfies the conditions $\mathbf{u}^*\cdot\mathbf{n} = 0$ on $\partial\Omega$ and $\operatorname{div}\mathbf{u}^* = 0$ in Ω and $\mathbf{f} \in L^1(0,T;\mathbf{W}^{4,2}(\Omega))$ then there exists $T_0 \in (0,T]$ and a unique solution $\mathbf{u}^0$ of the Euler problem* (1)–(3) *on the time interval* $(0,T_0)$ *such that* $\mathbf{u}^0 \in L^\infty(0,T_0;\mathbf{W}^{4,2}(\Omega))$.

Since the Euler equation in (1) *formally follows from the Navier–Stokes equations*

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = \nu\Delta\mathbf{u} + \mathbf{f}, \qquad \operatorname{div}\mathbf{u} = 0 \tag{4}$$

if the coefficient of viscosity ν tends to zero, there arises a natural question whether the solution $\mathbf{u}^0$ of the Euler problem is a limit for $\nu \to 0+$ of a (possibly unique) family of solutions $\mathbf{u}^\nu$ of an appropriate Navier–Stokes problem. By "appropriate" we mean the Navier–Stokes problem given by Eqs. (4)*, conditions* (2)*,* (3) *and completed by an appropriate additional boundary condition for velocity. Thus, we arrive at this problem:*

Problem (P). *To find a complementary boundary condition* (*) *for velocity, that enables us to extend the solution $\mathbf{u}^0$ of the Euler problem* (1)–(3) *to a branch $\{\mathbf{u}^\nu\}$ of strong solutions of the Navier–Stokes problem* (4), (2), (3) *and* (*)*, so that the branch is unique and continuous in dependence on ν in some interval* $[0,\nu^*)$.

In this Note, we deal with a solution of problem (P), provided that the complementary boundary condition (*) for velocity is the Navier-type condition

$$\mathbf{curl}\,\mathbf{u} \times \mathbf{n} = \mathbf{a}^\nu \quad \text{on } \partial\Omega \times (0,T_0), \tag{5}$$

where $\mathbf{a}^\nu$ is defined:

$$\mathbf{a}^\nu := \mathbf{curl}\big(\mathbf{u}^0 + \nu\mathbf{v}^0\big) \times \mathbf{n}. \tag{6}$$

(Here $\mathbf{u}^0$ is the solution of the Euler problem (1)–(3) and $\mathbf{v}^0$ solves the linear problem (8)–(10).)

An analogous result, with different boundary conditions, was recently proved in our paper [3]. As complementary boundary conditions for solutions of the Navier–Stokes equations, we used the two scalar condition $\mathbf{curl}\,\mathbf{u}^\nu\cdot\mathbf{n} = \mathbf{curl}(\mathbf{u}^0 + \nu\mathbf{v}^0)\cdot\mathbf{n}$ and $\mathbf{curl}^2\,\mathbf{u}^\nu\cdot\mathbf{n} = \mathbf{curl}^2(\mathbf{u}^0 + \nu\mathbf{v}^0)\cdot\mathbf{n}$, where $\mathbf{u}^0$ and $\mathbf{v}^0$ have the same meaning as in (6). (See [3].)

2. Structure of a solution to the Navier–Stokes problem (2)–(5) and main results

We construct a unique solution $\mathbf{u}^\nu$ of the Navier–Stokes problem (2)–(5) on the time interval $(0,T_0)$, that tends to the solution $\mathbf{u}^0$ of the Euler problem as $\nu \to 0+$, in the form

$$\mathbf{u}^\nu = \mathbf{u}^0 + \nu\mathbf{v}^0 + \nu\mathbf{w}^\nu \tag{7}$$

with an associated pressure $p^\nu = p^0 + \nu p^0_v + \nu p^\nu_w$. Functions $\mathbf{v}^0$ and p^0_v satisfy the linear problem

$$\partial_t \mathbf{v}^0 + \mathbf{u}^0\cdot\nabla\mathbf{v}^0 + \mathbf{v}^0\cdot\nabla\mathbf{u}^0 + \nabla p^0_v = \Delta\mathbf{u}^0, \qquad \operatorname{div}\mathbf{v}^0 = 0 \quad \text{in } Q_{T_0}, \tag{8}$$

$$\mathbf{v}^0(.,0) = \mathbf{0} \quad \text{in } \Omega, \tag{9}$$

$$\mathbf{v}^0\cdot\mathbf{n} = 0 \quad \text{on } \partial\Omega \times (0,T_0). \tag{10}$$

Then, substituting from (7) to (2)–(5), we observe that $\mathbf{w}^\nu$, p^ν_w must solve the nonlinear problem

$$\partial_t \mathbf{w}^\nu + \mathbf{u}^0 \cdot \nabla \mathbf{w}^\nu + \mathbf{w}^\nu \cdot \nabla \mathbf{u}^0 + \nu \mathbf{v}^0 \cdot \nabla \mathbf{w}^\nu + \nu \mathbf{w}^\nu \cdot \nabla \mathbf{v}^0 + \nu \mathbf{w}^\nu \cdot \nabla \mathbf{w}^\nu + \nu \mathbf{v}^0 \cdot \nabla \mathbf{v}^0 + \nabla p_w^\nu$$
$$= \nu \Delta \mathbf{w}^\nu + \nu \Delta \mathbf{v}^0, \qquad \operatorname{div} \mathbf{w}^\nu = 0 \quad \text{in } Q_{T_0}, \tag{11}$$
$$\mathbf{w}^\nu(.,0) = \mathbf{0} \quad \text{in } \Omega. \tag{12}$$

We construct a solution $\mathbf{w}^\nu$ of the problem (11)–(12) so that it satisfies the boundary conditions

$$\mathbf{w}^\nu \cdot \mathbf{n} = 0, \qquad \mathbf{curl}\, \mathbf{w}^\nu \times \mathbf{n} = \mathbf{0} \quad \text{on } \partial\Omega \times (0, T_0). \tag{13}$$

These conditions guarantee that function $\mathbf{u}^\nu$ satisfies the boundary conditions (3), (5).

Note that function $\mathbf{v}^0$ plays an important role in the structure of $\mathbf{u}^\nu$ because it "absorbs" the term $\nu \Delta \mathbf{u}^0$, which arises if we substitute for $\mathbf{u}^\nu$ from (7) to (4) – see the right-hand side of (8).

Below we list the main theorems. Theorems 2.1 and 2.2 are auxiliary and they concern the problems (8)–(10) and (11)–(13). Recall that $(0, T_0)$ is the time interval provided by [6] or [10], on which the solution $\mathbf{u}^0$ of the Euler problem exists.

Theorem 2.1 *(On the linear problem (8)–(10)). The problem* (8)–(10) *has a unique solution* $\mathbf{v}^0 \in L^\infty(0, T_0; \mathbf{W}^{2,2}_\sigma(\Omega))$. *The norm* $|||\mathbf{v}^0|||_{\infty;2,2}$ *depends on* Ω, T_0 *and on the norm* $|||\mathbf{u}^0|||_{1;4,2}$.

This theorem is proved in [3]. The principle of the proof is also explained in [5].

Theorem 2.2 *(On the nonlinear problem (11)–(13)). There exists* $\nu^* > 0$ *such that problem* (11)–(13) *has a unique solution* $\mathbf{w}^\nu \in L^\infty(0, T_0; \mathbf{W}^{1,2}_\sigma(\Omega)) \cap L^2(0, T_0; \mathbf{W}^{2,2}(\Omega))$ *for each* $\nu \in (0, \nu^*)$.

Solution $\mathbf{w}^\nu$ *depends continuously on* ν *in the norm* $|||.|||_{\infty;1,2} + |||.|||_{2;2,2}$.

There exist positive constants c_1, c_2 *and* c_3, *depending on* Ω, T_0 *and on the norms* $|||\mathbf{u}^0|||_{1;4,2}$ *and* $|||\mathbf{v}^0|||_{\infty;2,2}$, *however all independent of* ν, *so that*

$$|||\mathbf{w}^\nu|||_{\infty;0,2} \leqslant c_1 \nu, \qquad |||\mathbf{w}^\nu|||_{\infty;1,2} \leqslant c_2 \sqrt{\nu}, \qquad |||\mathbf{w}^\nu|||_{2;2,2} \leqslant c_3. \tag{14}$$

The idea of the proof of Theorem 2.2 is explained in Section 3. Theorems 2.1, 2.2 and formula (7) yield the next theorem, which represents the main result of the paper:

Theorem 2.3 *(On a family of solutions of the Euler or Navier–Stokes problem). Suppose that* $\mathbf{u}^* \in \mathbf{W}^{4,2}(\Omega) \cap \mathbf{L}^2_\sigma(\Omega)$ *and* $\mathbf{f} \in L^1(0, T; \mathbf{W}^{4,2}(\Omega))$. *Then there exists* $T_0 \in (0, T]$, $\nu^* > 0$ *and a unique family* $\{\mathbf{u}^\nu\}$ *(for* $\nu \in [0, \nu^*)$*) of solutions of the Euler problem* (1)–(3) *(if* $\nu = 0$*) or the Navier–Stokes problem* (2)–(5) *(if* $\nu \in (0, \nu^*)$*) in* $L^\infty(0, T_0; \mathbf{W}^{1,2}_\sigma(\Omega)) \cap L^2(0, T_0; \mathbf{W}^{2,2}(\Omega))$.

Solution $\mathbf{u}^\nu$ *depends continuously on* ν *in the norm* $|||.|||_{\infty;1,2} + |||.|||_{2;2,2}$.

Solution $\mathbf{u}^\nu$ *has the form* (7), *where* $\mathbf{v}^0$ *is the solution of the linear problem* (8)–(10) *and* $\mathbf{w}^\nu$ *is a solution of the nonlinear problem* (11)–(13). *The solutions* $\mathbf{v}^0$ *and* $\mathbf{w}^\nu$ *have the properties named in Theorems* 2.1 *and* 2.2.

3. Principles of the proof of Theorem 2.2

3.1. A special Stokes operator S

Let Γ_i, $1 \leqslant i \leqslant N$ be the components of $\partial\Omega$.

- $\mathbf{curl}_1$ denotes the operator **curl** defined in $D(\mathbf{curl}_1) := \{\mathbf{u} \in \mathbf{W}^{2,2}_\sigma(\Omega);\ \mathbf{curl}\,\mathbf{u} \times \mathbf{n} = \mathbf{0} \text{ on } \partial\Omega\}$.
- $\mathbf{curl}_2$ denotes the operator **curl** with the domain $D(\mathbf{curl}_2)$, which is the space of all divergence-free functions $\mathbf{u} \in \mathbf{W}^{1,2}(\Omega)$ such that $\mathbf{u} \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$ and $\mathbf{u}$ is orthogonal (in the scalar product of $\mathbf{L}^2(\Omega)$) to $\mathbf{G}(\Omega) := \{\nabla\varphi;\ \varphi \in C^\infty(\overline{\Omega}),\ \varphi = \text{const.} = C_i \text{ on } \Gamma_i \text{ for } i = 1, \dots, N\}$.
- We define $\mathbf{S} := \mathbf{curl}_2 \circ \mathbf{curl}_1$. $\mathbf{S}$ is a special Stokes operator in $\mathbf{L}^2_\sigma(\Omega)$. The domain of operator $\mathbf{S}$ is $D(\mathbf{S}) = D(\mathbf{curl}_1)$.

Further we list some properties of operators $\mathbf{curl}_1$, $\mathbf{curl}_2$ and $\mathbf{S}$:

- $\mathbf{curl}_1$ is a linear bijective operator from $D(\mathbf{curl}_1)$ onto $D(\mathbf{curl}_2)$. The inverse operator is bounded as an operator from $\mathbf{W}^{1,2}(\Omega)$ into $\mathbf{W}^{2,2}(\Omega)$.
- $\mathbf{curl}_2$ is a linear bijective operator from $D(\mathbf{curl}_2)$ onto $\mathbf{L}^2_\sigma(\Omega)$. The inverse operator is bounded as an operator from $\mathbf{L}^2(\Omega)$ into $\mathbf{W}^{1,2}(\Omega)$.
- Operator $\mathbf{S}$ is a linear bijective operator from $D(\mathbf{S})$ onto $\mathbf{L}^2_\sigma(\Omega)$. The inverse operator $\mathbf{S}^{-1}$ is bounded as an operator from $\mathbf{L}^2(\Omega)$ into $\mathbf{W}^{2,2}(\Omega)$. Note also that $\mathbf{S}$ commutes with the Helmholtz projector $\mathbf{P}_\sigma$ on $D(\mathbf{S})$. ($\mathbf{P}_\sigma$ is the orthogonal projector of $\mathbf{L}^2(\Omega)$ onto $\mathbf{L}^2_\sigma(\Omega)$.)
- $\mathbf{S}$ is a positive selfadjoint operator in $\mathbf{L}^2_\sigma(\Omega)$. Although $\mathbf{S}^{1/2}$ is different from $\mathbf{curl}_1$, one can easily check these identities for the norms: $\|\mathbf{S}^{1/2}\mathbf{u}\|_2^2 = (\mathbf{Su}, \mathbf{u})_2 = \|\,\mathbf{curl}\,\mathbf{u}\|_2^2 = \|\,\mathbf{curl}_1\,\mathbf{u}\|_2^2$ for $\mathbf{u} \in D(\mathbf{S})$. Consequently, $\|\mathbf{S}^{1/2}.\|_2$ is a norm in $D(\mathbf{S}^{1/2})$, equivalent to the norm $\|.\|_{1,2}$.

Using the Stokes operator, we can write $\nu\Delta\mathbf{w}^\nu = -\nu\mathbf{S}\mathbf{w}^\nu$ in Eq. (11).

3.2. A priori estimates of a solution of the problem (11)–(13)

The proof of Theorem 2.2 is mainly based on the next a priori estimates: At first we multiply Eq. (11) by $\mathbf{w}^\nu$ and integrate in Ω. We obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|\mathbf{w}^\nu\|_2^2 + \nu\|\mathbf{S}^{1/2}\mathbf{w}^\nu\|_2^2 = \int_\Omega \big[-\mathbf{w}^\nu \cdot \big(\nabla\mathbf{u}^0 + \nu\nabla\mathbf{v}^0\big)\cdot\mathbf{w}^\nu - \nu\mathbf{v}^0\cdot\nabla\mathbf{v}^0\cdot\mathbf{w}^\nu + \nu\mathbf{v}^0\cdot\mathbf{w}^\nu\big]\,\mathrm{d}\mathbf{x}.$$

Applying standard estimates to the right-hand side and integrating with respect to $t \in (0, T_0)$, we can derive:

$$|||\mathbf{w}^\nu|||_{\infty;0,2} \leqslant c_4\nu T_0 \mathrm{e}^{\nu T_0}. \tag{15}$$

As a second step, we multiply Eq. (11) by $\mathbf{S}\mathbf{w}^\nu$ and integrate in Ω. We get

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|\mathbf{S}^{1/2}\mathbf{w}^\nu\|_2^2 + \nu\|\mathbf{S}\mathbf{w}^\nu\|_2^2 = -\int_\Omega \big[\mathbf{u}^0\cdot\nabla\mathbf{w}^\nu + \mathbf{w}^\nu\cdot\nabla\mathbf{u}^0 + \nu\mathbf{v}^0\cdot\nabla\mathbf{w}^\nu + \nu\mathbf{w}^\nu\cdot\nabla\mathbf{v}^0\big]\cdot\mathbf{S}\mathbf{w}^\nu\,\mathrm{d}\mathbf{x}$$
$$- \nu\int_\Omega \big[\mathbf{w}^\nu\cdot\nabla\mathbf{w}^\nu + \mathbf{v}^0\cdot\nabla\mathbf{v}^0 - \Delta\mathbf{v}^0\big]\cdot\mathbf{S}\mathbf{w}^\nu\,\mathrm{d}\mathbf{x}.$$

The integral of $\nabla p_w^\nu \cdot \mathbf{S}\mathbf{w}^\nu$ equals zero because $\mathbf{S}\mathbf{w}^\nu = \mathbf{P}_\sigma\mathbf{S}\mathbf{w}^\nu \in \mathbf{L}^2_\sigma(\Omega)$. Applying nontrivial integration by parts, based on finer properties of operators $\mathbf{curl}_1$, $\mathbf{curl}_2$ and $\mathbf{S}$, we can estimate the right-hand side and arrive at the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{S}^{1/2}\mathbf{w}^\nu\|_2^2 + \nu\|\mathbf{S}\mathbf{w}^\nu\|_2^2 \leqslant c_5\big(\|\mathbf{u}^0\|_{4,2} + \nu\big)\|\mathbf{S}^{1/2}\mathbf{w}^\nu\|_2^2 + c_6\nu\|\mathbf{S}^{1/2}\mathbf{w}^\nu\|_2^6 + c_7\nu.$$

Integrating this inequality with respect to $t \in (0, T_0)$ and using (15), we deduce that there exists $\nu^* > 0$ and quantities $\mathfrak{S}(\nu, T_0)$, $\mathfrak{U}(\nu, T_0)$, depending also on $|||\mathbf{u}^0|||_{1;4.2}$, $|||\mathbf{v}^0|||_{\infty;2,2}$ and Ω, such that

$$\|\mathbf{S}^{1/2}\mathbf{w}^\nu(.,t)\|_2^2 \leqslant \nu\mathfrak{S}\big(\nu^*, T_0\big) \quad \text{and} \quad \int_0^{T_0}\|\mathbf{S}\mathbf{w}^\nu(.,t)\|_2^2\,\mathrm{d}t \leqslant \mathfrak{U}\big(\nu^*, T_0\big) \tag{16}$$

for $0 \leqslant t < T_0$ and $0 < \nu \leqslant \nu^*$. Estimates (15) and (16) imply inequalities (14) in Theorem 2.2.

4. Discussion

The question of a vanishing viscosity limit of solutions $\mathbf{u}^\nu$ of the Navier–Stokes equations belongs to important problems of fluid dynamics, whose only partial solution is so far known. The difficulties arise especially when the solutions $\mathbf{u}^\nu$ are required to satisfy the no-slip boundary condition $\mathbf{u}^\nu = \mathbf{0}$ on $\partial\Omega$. T. Kato [8] proved that solutions $\mathbf{u}^\nu$ of the Navier–Stokes equations with the no-slip boundary condition tend to a weak solution $\mathbf{u}^0$ of the Euler equations in the norm of $L^\infty(0, T'; \mathbf{L}^2(\Omega))$ if and only if the energy dissipation, corresponding to $\mathbf{u}^\nu$, in a boundary stripe of a width proportional to ν and in the time interval $(0, T')$ tends to zero as $\nu \to 0$.

The convergence of $\mathbf{u}^\nu$ to $\mathbf{u}^0$ in norms preserving the no-slip boundary condition cannot be generally expected because the solution $\mathbf{u}^0$ does not generally satisfy this boundary condition. The situation is, however, different if one considers solutions $\mathbf{u}^\nu$ of the Navier–Stokes equations with another boundary condition. Here we can mention e.g. the papers [1] by C. Bardos (the 2D case, a solution $\mathbf{u}^0 \in \mathbf{W}^{1,2}_\sigma(\Omega)$ of the Euler equation is constructed by solving the Navier–Stokes equations with the additional boundary condition $\nabla \times \mathbf{u}^\nu = \mathbf{0}$ and letting $\nu \to 0$) and [9] by A. Mahalov, B. Nicolaenko, C. Bardos and F. Golse (the case of a cylindrical domain $\Omega \in \mathbb{R}^3$, the authors found a family of the Navier–Stokes flows strongly convergent to a unique solution of the 3D Euler equation as $\nu \to 0+$, the initial velocity $\mathbf{u}^*$ is supposed to have a uniformly large vorticity and to satisfy the boundary conditions $\mathbf{curl}^j\, \mathbf{u}^* \cdot \mathbf{n} = 0$ on the envelope of the cylinder for $0 \leqslant j \leqslant s$, where s is at least 4).

Y.L. Xiao and Z.P. Xin [11] studied the zero-viscosity limit of a family of strong solutions of the Navier–Stokes problem (4), (2), (3) and (5) (with $\mathbf{a}^\nu = \mathbf{0}$). Their theory particularly implies that if $\mathbf{u}$ is a sufficiently smooth divergence-free vector function in Ω such that $\mathbf{u}^\nu \cdot \mathbf{n} = 0$, and $\mathbf{curl}\, \mathbf{u}^\nu \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$ then $\mathbf{curl}[\mathbf{curl}\, \mathbf{u}^\nu \times \mathbf{u}^\nu] \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$. However, this identity generally holds only on a flat part of the boundary, see a counterexample given by H. Beirão da Veiga and F. Crispo in [2].

The survey of a series of other related results is given in [5]. From the point of view of using the homogeneous Navier-type boundary conditions to solve problem (P), the previous papers [11], [2] and also the paper [7] by G.Q. Chen, D. Osborne and Z. Qian deserve a special attention.

Acknowledgements

The research was supported by the Northern Illinois University, by the University of Sud-Toulon-Var, by the Grant Agency of the Czech Republic (grant No. 201/08/0012) and by the research plan of the Ministry of Education of the Czech Republic No. MSM 6840770010.

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