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Large deviation inequalities for supermartingales and applications to directed polymers in a random environment

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Abstract

We first show exponential large deviation inequalities for supermartingales, which are optimal in the independent and identically distributed (iid) case. We then apply them to establish new exponential concentration inequalities for the free energy of directed polymers in random environment; as consequences we obtain upper bounds of its rate of convergences (in probability, almost surely, and in L^p), and give an expression for the free energy in terms of free energies of some multiplicative cascades. ***To cite this article: Q. Liu, F. Watbled, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Inégalités de grandes déviations pour les surmartingales et applications aux polymères dirigés en milieu aléatoire. Nous montrons d'abord des inégalités exponentielles de grandes déviations pour les surmartingales, qui sont optimales dans le cas de variables aléatoires indépendantes et identiquement distribuées (iid). Nous utilisons ensuite ce résultat pour établir des inégalités exponentielles de concentration pour l'énergie libre des polymères dirigés en milieu aléatoire ; nous en déduisons des bornes supérieures pour sa vitesse de convergence (en probabilité, presque sûrement, et dans L^p), et une expression de l'énergie libre en terme d'énergies libres de certaines cascades multiplicatives. ***Pour citer cet article : Q. Liu, F. Watbled, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Version française abrégée

Soit (S_n) une martingale adaptée à une filtration $(\mathcal{F}_n)$. Motivés par l'étude des polymères dirigés en milieu aléatoire, nous nous intéressons à des inégalités exponentielles du type :

$$P\big[|S_n| > nx\big] = O\big(e^{-c(x)n}\big), \tag{1}$$

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pour $x > 0$ et $c(x) > 0$. Quand les différences $X_i = S_i - S_{i-1}$ sont indépendantes, identiquement distribuées (iid), et d'espérance nulle, il est bien connu que (1) a lieu si et seulement si il existe $\delta > 0$ tel que (voir [7, p. 137]) :

$$\mathbb{E}e^{\delta|X_1|} < \infty. \tag{2}$$

Pour une suite de différences de martingales, Lesigne et Volný [4] ont montré que si

$$\mathbb{E}e^{|X_i|} \leqslant K \quad \text{pour tout } i, \tag{3}$$

alors pour tout $x > 0$,

$$P\left[\frac{S_n}{n} > x\right] = O\left(e^{-\frac{1}{4}x^{2/3}n^{1/3}}\right); \tag{4}$$

ils ont aussi montré que l'exposant $n^{1/3}$ est optimal sous l'hypothèse de Cramer (2). Dans le théorème suivant nous montrons que la conclusion (1) reste valable pour les martingales si on remplace la condition de Cramer (2) non-conditionnelle par une condition de Cramer conditionnelle ; par ailleurs le résultat est aussi valable pour les surmartingales. Plus précisément, nous avons le résultat suivant :

Théorème 0.1. *Soit (X_i) $(1 \leqslant i \leqslant n)$ une suite de différences de surmartingales adaptée à une filtration $(\mathcal{F}_i)$ (c'est-dire que pour tout $i \in [1, n]$, X_i est $\mathcal{F}_i$-mesurable et $\mathbb{E}(X_i|\mathcal{F}_{i-1}) \leqslant 0$; par convention $F_0 = \{\emptyset, \Omega\}$). On suppose qu'il existe des constantes $Q \geqslant 1$, $\delta > 0$, $K_i > 0$ telles que pour tout $i \in [1, n]$,*

$$\mathbb{E}\left[e^{\delta|X_i|^Q} \,\middle|\, \mathcal{F}_{i-1}\right] \leqslant K_i \quad p.s. \tag{5}$$

Soit $K \geqslant \frac{K_1+\cdots+K_n}{n}$. Alors il existe une constante $c > 0$ dépendant uniquement de Q, δ et K, telle que :

$$P\left[\frac{S_n}{n} > x\right] \leqslant \begin{cases} e^{-ncx^2} & si\ x \in]0, 1], \\ e^{-ncx^Q} & si\ x \in]1, \infty[. \end{cases} \tag{6}$$

Réciproquement, si les X_i sont iid avec $EX_i \leqslant 0$, et si $P[\frac{S_n}{n} > x] \leqslant e^{-ncx^Q}$ pour un entier $n \geqslant 1$, certaines constantes $Q \geqslant 1$, $c > 0$, $x_1 > 0$, et tout $x \geqslant x_1$, alors pour tout $\delta \in]0, c[$, il existe $K = K(\delta, Q, c, x_1) > 0$ tel que

$$\mathbb{E}\left[e^{\delta X_1^{+Q}}\right] \leqslant K, \quad où\ X_1^+ = \max(X_1, 0). \tag{7}$$

Le cas $Q = 1$ fournit une généralisation de l'inégalité de Bernstein sur les suites iid ; le cas $Q = 2$ donne une extension de l'inégalité de Hoeffding sur les martingales d'accroissements bornés au cas d'une surmartingale d'accroissements non forcément bornés.

Cette inégalité nous permet d'obtenir de nouveaux résultats sur le modèle d'un polymère dirigé en environnement aléatoire, décrit par exemple dans [1–3], où on s'intéresse au comportement asymptotique de la fonction de partition normalisée W_n, et de l'énergie libre $\frac{1}{n}\ln W_n$. En décomposant l'énergie centrée $\ln W_n - \mathbb{Q}[\ln W_n]$ comme une suite de différences de martingales (comme dans [2]) et en appliquant l'inégalité précédente, nous obtenons une inégalité de concentration exponentielle qui n'était connue que dans le cas d'un environnement gaussien ou borné. Comme conséquences, nous montrons que : (a) l'énergie libre $\frac{\ln W_n}{n}$ converge en probabilité avec une vitesse exponentielle, (b) elle converge p.s. et dans L^p avec une vitesse $O(\sqrt{\frac{\ln n}{n}})$, (c) sa limite est égale à l'infimum des énergies libres de certaines cascades généralisées de Mandelbrot (cf. [5]), ce qui transforme une inégalité de Comets et Vargas [3] en une *égalité*.

1. Introduction

Our work was initially motivated by the study of the free energy of a directed polymer in a random environment. Comets and Vargas [3] proved that the free energy (at ∞) is bounded by the infimum of those of some generalized multiplicative cascades, and that the equality holds if the environment is bounded or Gaussian. The essential point in their proof for the equality is an exponential concentration inequality for the free energy, which was not known for a general environment. Using a large deviation inequality of Lesigne and Volný [4] on martingales, Comets, Shiga and Yoshida [2] did obtain a concentration inequality for the free energy; but their bound is larger than the exponential one, and is not sharp enough to imply the equality mentioned above. Another non-satisfactory point of their inequality is that it cannot be used to prove rigorous results on the rate of almost sure (a.s.) or L^p convergence of the free energies.

In this Note, we first show new exponential inequalities for finite sequences of supermartingale differences, which are also new for martingale differences. We then show how to use them to obtain exponential concentration inequalities for the free energy of a directed polymer in general random environment. As corollaries, we get upper bounds for the rate of convergences of the free energy, and improve the inequality of Comets and Vargas [3] mentioned above to an equality. Complete proofs will be found in [6].

2. Exponential inequalities for supermartingales

In this section, we show new exponential large deviation inequalities for supermartingales. In particular, we give an extension of Bernstein and Hoeffding's inequalities to supermartingales with unbounded differences. Our results are sharp even in the iid case.

Let $(X_i)_{1\leqslant i\leqslant n}$ be a sequence of real-valued supermartingale differences defined on a probability space $(\Omega, \mathcal{F}, P)$, adapted to a filtration $(\mathcal{F}_i)$, with $\mathcal{F}_0 = \{\emptyset, \Omega\}$. This means that for each (integer) $1 \leqslant i \leqslant n$, X_i is $\mathcal{F}_i$-measurable and $\mathbb{E}[X_i|\mathcal{F}_{i-1}] \leqslant 0$ a.s. We are interested in the convergence rate of the large deviation probabilities $P[S_n/n > x]$.

Theorem 2.1. *Let $(X_i)_{1\leqslant i\leqslant n}$ be a finite sequence of supermartingale differences with respect to a filtration $(\mathcal{F}_i)_{1\leqslant i\leqslant n}$. Assume that for some constants $Q \geqslant 1$, $R > 0$ and $K_i > 0$ such that for all $i \in [1, n]$,*

$$\mathbb{E}\big[e^{R|X_i|^Q} \bigm| \mathcal{F}_{i-1}\big] \leqslant K_i \quad a.s. \tag{8}$$

Let $K \geqslant \frac{K_1+\cdots+K_n}{n}$. Then there exists a constant $c > 0$ depending only on K, Q, R, such that

$$P\left[\frac{S_n}{n} > x\right] \leqslant \begin{cases} \exp(-ncx^2) & \text{if } x \in\,]0, 1], \\ \exp(-ncx^Q) & \text{if } x \in\,]1, \infty[. \end{cases} \tag{9}$$

Conversely, if (X_i) are iid with $EX_i \leqslant 0$ and if $P[\frac{S_n}{n} > x] \leqslant \exp(-ncx^Q)$ for some $n \geqslant 1$, $c > 0$, $Q \geqslant 1$, $x_1 > 0$ and all $x \geqslant x_1$, then for all $R \in\,]0, c[$,

$$\mathbb{E}\big[e^{RX_1^{+Q}}\big] \leqslant K, \quad \text{where } X_1^+ = \max(X_1, 0) \text{ and } K = e^{Rx_1^Q} + \frac{R}{c-R} e^{-(c-R)x_1^Q}. \tag{10}$$

Moreover, if (8) holds for $Q = 1$ and $R = 1$, then

$$P\left[\frac{S_n}{n} > x\right] \leqslant \exp\big(-n\big(\sqrt{x+K} - \sqrt{K}\big)^2\big) \quad \forall x > 0. \tag{11}$$

The case $Q = 1$ is an extension of Bernstein's inequality (cf. [7, p. 57]) on independent sequences to supermartingales. Notice that (11) implies (9) with $Q = 1$. Its proof relies on the following two keypoints:

(a) If $\mathbb{E}[X] \leqslant 0$ and $\mathbb{E}[e^{|X|}] \leqslant K$ then for all $t \in\,]0, 1[$, $\mathbb{E}[e^{tX}] \leqslant \exp(\frac{Kt^2}{1-t})$;
(b) if $\mathbb{E}[e^{tX_i}|\mathcal{F}_{i-1}] \leqslant e^{l_i(t)}$ a.s. for all i, then $\mathbb{E}[e^{tS_n}] \leqslant \exp(\sum_{i=1}^n l_i(t))$.

The submultiplicativity (b) for a $(\mathcal{F}_i)$-adapted sequence (X_i) corresponds to the multiplicativity $\mathbb{E}[e^{tS_n}] = \prod_{i=1}^n \mathbb{E}[e^{tX_i}]$ for an independent sequence (X_i). This explains why it is natural to consider the conditional Laplace transform $\mathbb{E}[e^{tX_i}|\mathcal{F}_{i-1}]$ in the supermartingale case, instead of the Laplace transform $\mathbb{E}[e^{tX_i}]$ in the independent case.

The case $Q = 2$ extends the following well-known Hoeffding's inequality (also called Azuma's inequality in the literature, although it was first found by Hoeffding): if (X_k) is a sequence of martingale differences with $|X_k| \leqslant a$ a.s. for some constant $a \in\,]0, \infty[$, then for where $c = 1/(2a^2)$, all $n \geqslant 1$ and all $x > 0$,

$$P\left[\pm\frac{S_n}{n} > x\right] \leqslant e^{-ncx^2}. \tag{12}$$

In fact, by our result for $Q = 2$, we obtain:

Corollary 2.2 *(Extension of Hoeffding's inequality). When (X_k) are iid, then there is a constant $c > 0$ such that* (12) *holds for all $n \geqslant 1$ and all $x > 0$, if and only if for some $\delta > 0$,*

$$\mathbb{E}e^{\delta X_1^2} < \infty. \tag{13}$$

As a direct consequence of Theorem 2.1 we have the following estimation for the rate of a.s. and L^p convergences. (For this kind of results the moment condition can certainly be relaxed.)

Corollary 2.3. *Let $(X_i)_{1 \leqslant i \leqslant n}$ be a sequence of supermartingale differences. Assume that for all $i \in [1, n]$ and some constants $K_i > 0$,*

$$\mathbb{E}\big[e^{|X_i|} \bigm| \mathcal{F}_{i-1}\big] \leqslant K_i \quad a.s. \tag{14}$$

Let $K = \limsup_{n \to +\infty} \frac{K_1 + \cdots + K_n}{n}$ and $S_n^+ = \max(0, S_n)$. Then

$$\limsup_{n \to +\infty} \frac{S_n^+}{\sqrt{n \ln n}} \leqslant 2\sqrt{K} \quad a.s., \tag{15}$$

and

$$\limsup_{n \to +\infty} n^{\frac{p}{2}} \mathbb{E}\left[\left(\frac{S_n^+}{n}\right)^p\right] \leqslant p 2^{p-1} K^{\frac{p}{2}} \Gamma\left(\frac{p}{2}\right) \quad \forall p > 0. \tag{16}$$

3. Concentration inequalities for directed polymers in random environment

We now apply our inequalities on martingales to obtain exponential concentration inequalities for directed polymers in random environment. Let $\omega = (\omega_n)_{n \in \mathbb{N}}$ be the simple random walk on the d-dimensional integer lattice $\mathbb{Z}^d$ starting at 0, defined on a probability space $(\Omega, \mathcal{F}, P)$. Let $(\eta(n, x))_{(n,x) \in \mathbb{N} \times \mathbb{Z}^d}$ be a sequence of real valued, non-constant and i.i.d. random variables defined on another probability space $(E, \mathcal{E}, \mathbb{Q})$. Here ω represents the directed polymer, $(\eta(n, x))$ the random environment. The expectation with respect to P and $\mathbb{Q}$ will also be denoted by P and $\mathbb{Q}$ respectively. Let $\eta = \eta(0, 0)$ and $\lambda(\beta) = \ln \mathbb{Q}[e^{\beta\eta}]$ be its logarithmic moment generating function. We fix $\beta > 0$ (otherwise we consider $-\eta$) and assume only $\lambda(\pm\beta) < \infty$, which is equivalent to $\mathbb{Q}[e^{\beta|\eta|}] < \infty$. We are interested in the asymptotic behavior of the normalized partition function:

$$W_n(\beta) = P\big[\exp\big(\beta H_n - n\lambda(\beta)\big)\big], \quad \text{with } H_n(\omega) = \sum_{j=1}^{n} \eta(j, \omega_j), \tag{17}$$

and the free energy $\frac{1}{n} \ln W_n(\beta)$. For simplicity, we shall write W_n for $W_n(\beta)$. We have:

Theorem 3.1. *If $K := 2\exp(\lambda(-\beta) + \lambda(\beta)) < \infty$, then for all $n \geqslant 1$,*

$$\mathbb{Q}\left[\pm\frac{1}{n}\big(\ln W_n - \mathbb{Q}[\ln W_n]\big) > x\right] \leqslant \exp\big(-n\big(\sqrt{x + K} - \sqrt{K}\big)^2\big) \quad \forall x > 0. \tag{18}$$

More generally, if for some constants $Q \geqslant 1$, $R > 0$ and $K_0 > 0$, $\mathbb{Q}[e^{R|\eta|^Q}] \leqslant K_0$, then for some constant $c = c(Q, R, K_0) > 0$ and all $n \geqslant 1$,

$$\mathbb{Q}\left[\pm\frac{1}{n}\big|\ln W_n - \mathbb{Q}[\ln W_n]\big| > x\right] \leqslant \begin{cases} e^{-ncx^2} & \text{if } x \in \,]0, 1], \\ e^{-ncx^Q} & \text{if } x \in \,]1, \infty[. \end{cases} \tag{19}$$

For the proof, as in [2], we write $\ln W_n - \mathbb{Q}[\ln W_n]$ as a sum of $(\mathcal{E}_j)_{1 \leqslant j \leqslant n}$ martingale differences:

$$\ln W_n - \mathbb{Q}[\ln W_n] = \sum_{j=1}^{n} V_{n,j}, \quad \text{with } V_{n,j} = \mathbb{Q}_j[\ln W_n] - \mathbb{Q}_{j-1}[\ln W_n], \tag{20}$$

where $\mathbb{Q}_j$ denotes the conditional expectation with respect to $\mathbb{Q}$, given $\mathcal{E}_j = \sigma[\eta(i,x)\colon\ 1 \leqslant i \leqslant j,\ x \in \mathbb{Z}^d]$. With this decomposition, Theorem 3.1 is a direct consequence of Theorem 2.1 and the fact that

$$\mathbb{Q}_{j-1}\big[\exp\big(|V_{n,j}|\big)\big] \leqslant K := 2\exp\big(\lambda(\beta) + \lambda(-\beta)\big). \tag{21}$$

Using again the inequality (21), by Corollary 2.3 we obtain immediately:

Corollary 3.2. *If $K := 2\exp(\lambda(-\beta) + \lambda(\beta)) < \infty$, then*

$$\frac{1}{n}\ln W_n - \frac{1}{n}\mathbb{Q}[\ln W_n] \to 0 \quad \text{a.s. and in } L^p, \tag{22}$$

with

$$\limsup_{n\to+\infty} \sqrt{\frac{n}{\ln n}} \left| \frac{\ln W_n}{n} - \frac{\mathbb{Q}[\ln W_n]}{n} \right| \leqslant 2\sqrt{K} \quad \text{a.s.}, \tag{23}$$

and

$$\limsup_{n\to+\infty} n^{\frac{p}{2}} \mathbb{Q}\left[\left|\frac{\ln W_n - \mathbb{Q}[\ln W_n]}{n}\right|^p\right] \leqslant p2^p K^{\frac{p}{2}} \Gamma\left(\frac{p}{2}\right) \quad \forall p > 0. \tag{24}$$

4. Consequences on asymptotic properties of the free energy

It is well known that the sequence $\mathbb{Q}[\ln W_n]$ is superadditive, hence the following limit exists:

$$p_-(\beta) = \lim_{n\to\infty} \frac{1}{n}\mathbb{Q}\big[\ln(W_n)\big] = \sup_n \frac{1}{n}\mathbb{Q}\big[\ln(W_n)\big] \in \,]-\infty, 0]. \tag{25}$$

As an immediate consequence of (25) and (22), we see that if $\mathbb{Q}[e^{\beta|\eta|}] < +\infty$, then

$$p_-(\beta) = \lim_{n\to\infty} \frac{1}{n}\ln(W_n) \in \big[\beta\mathbb{Q}[\eta] - \lambda(\beta), 0\big] \quad Q\text{-a.s. and in } L^p, \forall p \geqslant 1. \tag{26}$$

The a.s. convergence was proved in [2] under the moment condition that $Q[e^{3\beta|\eta|}] < \infty$; the case where η has no finite exponential moment was considered in [8]. Here we show results about the rate of convergence (in probability, a.s. and in L^p), as applications of Theorem 3.1 and Corollary 3.2.

Theorem 4.1. *If $K := 2\exp(\lambda(-\beta) + \lambda(\beta)) < \infty$, then $\forall \delta \in]0,1[$, $\forall x > 0$, there exists $n_0 = n_0(\delta, x) > 0$ such that $\forall n \geqslant n_0$,*

$$\mathbb{Q}\left[\left|\frac{1}{n}\ln W_n - p_-(\beta)\right| > x\right] \leqslant 2\exp\big(-n\big(\sqrt{(1-\delta)x + K} - \sqrt{K}\big)^2\big). \tag{27}$$

Using Lesigne and Volný's martingale inequality (4), Comets, Shiga and Yoshida [2] proved that if $\mathbb{Q}[e^{\beta|\eta|}] < +\infty$ for all $\beta > 0$, then for all $x > 0$, there exists $n_0 \in \mathbb{N}^*$ such that for any $n \geqslant n_0$,

$$\mathbb{Q}\left[\left|\frac{1}{n}\ln W_n - \frac{1}{n}\mathbb{Q}[\ln W_n]\right| > x\right] \leqslant \exp\left(-\frac{n^{\frac{1}{3}}x^{\frac{2}{3}}}{4}\right). \tag{28}$$

Our result is sharper as $n^{1/3}$ is replaced by n; this improvement is essential to show an expression of the free energy in terms of those of some multiplicative cascades (cf. Theorem 4.3). Another advantage is that our conclusion holds for all n, not only for n large enough; this enables us to show a.s. and L^p convergences rates (cf. Corollary 3.2).

Corollary 4.2. *If $K := 2\exp(\lambda(-\beta) + \lambda(\beta)) < \infty$, then*

$$\limsup_{n\to+\infty} \sqrt{\frac{n}{\ln n}} \left| \frac{\ln W_n}{n} - p_-(\beta) \right| \leqslant 2\sqrt{K}\big(1 + \sqrt{d}\big) \quad \text{a.s.}, \tag{29}$$

and

$$\limsup_{n\to+\infty} \sqrt{\frac{n}{\ln n}} \left\| \frac{\ln W_n}{n} - p_-(\beta) \right\|_p \leqslant 2\sqrt{Kd} \quad \forall p \geqslant 1. \tag{30}$$

In [3], Comets and Vargas introduced, for each $m \geqslant 1$, a generalized multiplicative cascade (cf. [5]) $(W_{m,n}^{tree})_{n\geqslant 1}$ associated to the random vector $(W_m(0,x))_{x\in L_m}$, where

$$W_m(0,x) = P\big[\exp\big(\beta H_m(\omega) - m\lambda(\beta)\big);\ \omega_m = x\big], \quad \text{and} \quad L_m = \big\{x \in \mathbb{Z}^d,\ P(\omega_m = x) > 0\big\}; \tag{31}$$

its free energy is

$$p_m^{tree}(\beta) = \inf_{\theta\in]0,1]} v_m(\theta), \quad \text{with } v_m(\theta) = \frac{1}{\theta}\ln\left(\mathbb{Q}\left[\sum_{x\in L_m} W_m(0,x)^\theta\right]\right). \tag{32}$$

They proved that $p_-(\beta) \leqslant \inf_{m\geqslant 1} \frac{1}{m} p_m^{tree}(\beta) = \lim_{m\to+\infty} \frac{1}{m} p_m^{tree}(\beta)$, and that the inequality can be replaced by an equality if the environment is Gaussian or bounded. By Theorem 3.1, we can prove that the equality also holds for a general environment:

Theorem 4.3. *If* $\mathbb{Q}[e^{\beta|\eta|}] < +\infty$, *then*

$$p_-(\beta) = \inf_{m\geqslant 1} \frac{1}{m} p_m^{tree}(\beta). \tag{33}$$

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