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# Optimal logarithmic estimates in Hardy–Sobolev spaces $H^{k,\infty}$

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**Abstract**

We prove sharp logarithmic estimates of optimal type in the Hardy–Sobolev spaces $H^{k,\infty}$ ($k \in \mathbb{N}^*$), thus extending earlier cases. These estimations are used in particular to establish logarithmic stability results for the Cauchy problem and the inverse problem of the identification of Robin's coefficient by boundary measurements. ***To cite this article: S. Chaabane, I. Feki, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***


**Résumé**

**Estimations logarithmiques optimales dans les espaces de Hardy–Sobolev $H^{k,\infty}$.** On montre des résultats de stabilité logarithmique de type optimal dans les espaces de Hardy–Sobolev $H^{k,\infty}$ ($k \in \mathbb{N}^*$). Ces estimations s'avèrent comme une extension des résultats déjà établis, et seront utilisées en particulier pour établir des résultats de stabilité logarithmique du problème de Cauchy et du problème inverse d'identification du coefficient de Robin par des mesures de surface. ***Pour citer cet article : S. Chaabane, I. Feki, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***


## Version française abrégée

Soient $\mathbb{D}$ le disque unité du plan complexe $\mathbb{C}$ et $I$ un ouvert connexe du cercle unité $\mathbb{T}$ de longueur $2\pi\lambda$ ; $\lambda \in ]0,1[$. Soient $\mu$ la mesure de Lebesgue sur $\mathbb{T}$ et $\|\cdot\|_{L^1(I)}$ la norme $L^1$ sur $I$ définie pour toute fonction mesurable $f : I \mapsto \mathbb{C}$ par :

$$\|f\|_{L^1(I)} = \frac{1}{2\pi\lambda}\int_I |f|\,\mathrm{d}\mu.$$

Pour tout ouvert non vide $J$ de $\mathbb{T}$ et pour tout $(k,p) \in \mathbb{N}^* \times [1,+\infty]$, on notera par $\|\cdot\|_{W^{k,p}(J)}$ la norme usuelle de l'espace de Sobolev $W^{k,p}(J)$. On désigne par $H^\infty$ l'ensemble des fonctions analytiques bornées sur $\mathbb{D}$ et par

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$H^{k,\infty} = \{f \in H^\infty,\ f^{(j)} \in H^\infty,\ j = 0, \ldots, k\}$ l'espace de Hardy–Sobolev du disque unité $\mathbb{D}$, où $f^{(j)}$ désigne la dérivée complexe d'ordre $j$ de $f$. On munit l'espace $H^{k,\infty}$ de la norme usuelle :

$$\|f\|_{H^{k,\infty}} = \max_{0 \leqslant j \leqslant k} \left(\|f^{(j)}\|_{L^\infty(\mathbb{T})}\right).$$

Le résultat essentiel de cette Note consiste à établir (Theorem 2.4) une estimation logarithmique permettant de contrôler la norme $\|f\|_{L^\infty(\mathbb{T})}$ en fonction de la norme $\|f\|_{L^1(I)}$. Plus précisément, nous démontrons l'inégalité $\|f\|_{L^\infty(\mathbb{T})} \leqslant \frac{8/(1-\frac{1}{2e})}{|\lambda \log(\|f\|_{L^1(I)})|}$ pour tout $f \in \mathcal{B}_{1,\infty}$, où pour $k \in \mathbb{N}$, $\mathcal{B}_{k,\infty}$ désigne la boule unité fermée de $H^{k,\infty}$.

Remarquons ici que cette estimation reste valable dans le cas d'un domaine simplement connexe de $\mathbb{R}^2$ de frontière une courbe de Jordan de classe $\mathcal{C}^{1,\beta}$ pour un certain $\beta \in ]0, 1[$.

L'équation (3) du Theorem 2.4, montre qu'il est impossible de trouver une fonction $\varepsilon$ qui tend vers 0 en 0, telle que $\|f\|_{L^\infty(\mathbb{T})} \leqslant \frac{1}{|\log(\|f\|_{L^1(I)})|}\varepsilon(\|f\|_{L^1(I)})$ pour tout $f \in \mathcal{B}_{1,\infty}$, ce qui prouve que l'estimation (2) est de type optimal. Une question ouverte, consiste à chercher la constante optimale $C = \sup_{g \in \mathcal{B}_{1,\infty}} \|g\|_{L^\infty(\mathbb{T})} |\log(\|g\|_{L^1(I)})|$.

On étend ensuite les résultats du Theorem 2.4 au cas des $H^{k,\infty}$.

Les premières estimations de ce type remontent à L. Baratchart et M. Zerner [1] qui ont établi un résultat de contrôle de type $\frac{\log\log}{\log}$ en norme $L^2$ dans le cas des espaces de Hardy $H^2$ du disque unité $\mathbb{D}$. Un résultat similaire en norme $L^2$ a été démontré par Leblond et al. [6] dans le cas d'une couronne $\mathcal{G} = \mathbb{D} \setminus s\mathbb{D}$ ; $0 < s < 1$. Il s'agit de contrôler la norme $L^2$ du bord intérieur $s\mathbb{T}$ d'une fonction $g$ suffisament régulière par celle de sa norme $L^2$ du bord extérieur $\mathbb{T}$. Ce résultat a été utilisé essentiellement pour établir des estimations de stabilité logarithmique du problème inverse d'identification du paramètre de Robin généralisant les résultats de [1] et [3]. La méthode utilisée n'est pas applicable dans notre cas, puisqu'elle se base sur la structure Hilbertienne de l'espace de Hardy $H^2$.

Comme applications, nous établissons dans la dernière section de cette Note un résultat de stabilité logarithmique de type optimal du problème de Cauchy et un autre de type $\frac{1}{\log}$ en norme $L^\infty$ du problème inverse d'identification du paramètre de Robin par des mesures de surface améliorant ainsi le résultat de stabilité de type $(\frac{\log\log}{\log})^a$ $(a < \frac{1}{2})$, démontré dans [3].

## 1. Introduction

Let $\mathbb{D}$ be the open unit disk of $\mathbb{C}$ and $I$ any connected open subset of the unit circle $\mathbb{T}$. We denote by $\mu$ the Lebesgue measure of $\mathbb{T}$ and by $\lambda = \frac{\mu(I)}{2\pi}$. We assume further that $\lambda \in ]0, 1[$. For every non-empty connected open subset $J$ of $\mathbb{T}$ and for every $(k, p) \in \mathbb{N}^* \times [1, +\infty]$, we designate by $\|\cdot\|_{W^{k,p}(J)}$ the usual norm of the Sobolev space $W^{k,p}(J)$. We denote by $H^\infty$ the space of bounded analytic functions on $\mathbb{D}$ and for any $k \in \mathbb{N}^*$, we designate by $H^{k,\infty} = \{f \in H^\infty,\ f^{(j)} \in H^\infty,\ j = 0, \ldots, k\}$, the Hardy–Sobolev space of $\mathbb{D}$, where $f^{(j)}$ denotes the $j$th complex derivative of $f$. We endow $H^{k,\infty}$ with the usual norm:

$$\|f\|_{H^{k,\infty}} = \max_{0 \leqslant j \leqslant k} \left(\|f^{(j)}\|_{L^\infty(\mathbb{T})}\right).$$

Let $\mathcal{B}_{k,\infty}$ be the closed unit ball of $H^{k,\infty}$: $\mathcal{B}_{k,\infty} = \{f \in H^{k,\infty},\ \|f\|_{H^{k,\infty}} \leqslant 1\}$.

The present Note aims to establish sharp optimal logarithmic estimates in the Hardy–Sobolev spaces $H^{k,\infty}$ of $\mathbb{D}$ for any $k \in \mathbb{N}^*$. These results extend then earlier cases [1] and [3] and allow us in particular to establish logarithmic stability results for the Cauchy problem with a Laplace operator and to prove an error estimate for the inverse problem of the identification of a Robin's coefficient improving thus the results of [3] and [4].

In the case of an annulus $\mathcal{G} = \mathbb{D} \setminus s\mathbb{D}$; $0 < s < 1$, Leblond et al. [6], proved a similar result with respect to the $L^2$ norm. Their methods are based on the Hilbertian properties of the Hardy space $H^2$ and provides logarithmic estimates about function's behaviour on $s\mathbb{T}$ from its behaviour on $\mathbb{T}$. These estimates are interestingly used to prove logarithmic stability results of the Robin's inverse problem which can be viewed as an extension of the result of [1] and [3].

## 2. Optimal logarithmic estimates in $H^{k,\infty}$

We start this section by recording a variant of the Hardy–Landau–Littlewood inequality (see [2] for more details) which will play an important role in the proof of our main results.

**Lemma 2.1.** *Let $j \in \mathbb{N}$ such that $j \geqslant 2$. Then, there exists a non-negative constant $C_\infty(j)$ such that for all $g \in H^{j,\infty}$, we have*:

$$\|g'\|_{L^\infty(\mathbb{T})} \leqslant C_\infty(j)\|g^{(j)}\|^{1/j}_{L^\infty(\mathbb{T})}\|g\|^{1-1/j}_{L^\infty(\mathbb{T})}. \tag{1}$$

*Moreover, $C_\infty(2) = \sqrt{2}$, and the other constants $C_\infty(j)$ could be inductively estimated.*

We adapt the same arguments developed in [1, Lemma 4.1] with slight shifts to prove the following:

**Lemma 2.2.** *Let $g \in H^\infty$ and $m = \|g\|_{L^\infty(\mathbb{T})}$. Then, for every $z \in \overline{\mathbb{D}}$, we have*:

$$\big|g(z)\big| \leqslant m^{1-\frac{\lambda}{2}(1-|z|)}\|g\|^{\frac{\lambda}{2}(1-|z|)}_{L^1(I)}.$$

Next, we establish the following control lemma:

**Lemma 2.3.** *Let $f \in H^{1,\infty}$ and $m = \|f\|_{L^\infty(\mathbb{T})}$. We suppose that $f$ is not identically zero and we denote by $F$ the primitive of $f$ vanishing at* 0*. Then, for every $z \in \overline{\mathbb{D}}$, we have*:

$$\big|F(z)\big| \leqslant \frac{m}{|\frac{\lambda}{2}\log(\|\frac{f}{m}\|_{L^1(I)})|}.$$

We are now in a position to prove our main result concerning uniform norm estimates in $H^{1,\infty}$.

**Theorem 2.4.** *Let $f \in \mathcal{B}_{1,\infty}$, then*:

$$\|f\|_{L^\infty(\mathbb{T})} \leqslant \frac{8/(1-\frac{1}{2e})}{|\lambda\log(\|f\|_{L^1(I)})|}. \tag{2}$$

*Moreover, for $I = \{e^{i\theta},\ \frac{\pi}{2} \leqslant \theta \leqslant \frac{3\pi}{2}\}$, there exists a sequence of functions $f_n \in \mathcal{B}_{1,\infty}$ such that*:

$$\lim_{n\to+\infty}\|f_n\|_{L^\infty(\mathbb{T})}\big|\log\big(\|f_n\|_{L^1(I)}\big)\big| \geqslant 1. \tag{3}$$

**Proof.** Let $F$ be the primitive of $f$ such that $F(0) = 0$ and $m \geqslant \max(\|f\|_{L^\infty(\mathbb{T})}, 1)$.

According to Lemma 2.3, we have:

$$\big|F(z)\big| \leqslant \frac{m}{|\frac{\lambda}{2}\log(\|\frac{f}{m}\|_{L^1(I)})|}. \tag{4}$$

We get therefore from [5, Theorem 3.11] and the Hardy–Landau–Littlewood inequality (1), that:

$$\|f\|_{L^\infty(\mathbb{T})} \leqslant 2\|F\|^{1/2}_{L^\infty(\mathbb{T})},$$

and consequently

$$\|f\|_{L^\infty(\mathbb{T})} \leqslant m_1 := 2\left(\frac{m}{|\frac{\lambda}{2}\log(\|\frac{f}{m}\|_{L^1(I)})|}\right)^{1/2}. \tag{5}$$

Making use of (4) and (5) for the new estimate $m_1$ of $\|f\|_{L^\infty(\mathbb{T})}$, one obtains:

$$\|f\|_{L^\infty(\mathbb{T})} \leqslant 2\left(\frac{m_1}{|\frac{\lambda}{2}\log(\|\frac{f}{m_1}\|_{L^1(I)})|}\right)^{1/2}. \tag{6}$$

Let $\eta(x) = x|\log x|^{1/2}$ and $\alpha = 1 - \frac{1}{2e}$. Since $m \geqslant 1$ and $\eta(x) \leqslant x^\alpha$ in $]0, 1]$, we get:

$$\left\|\frac{f}{m_1}\right\|_{L^1(I)} = \frac{(m\frac{\lambda}{2})^{1/2}}{2}\eta\left(\left\|\frac{f}{m}\right\|_{L^1(I)}\right) \leqslant \|f\|^\alpha_{L^1(I)}.$$

From (6) and the monotonicity of the mapping $\varepsilon(x) = \frac{1}{|\log x|}$, we obtain:

$$\|f\|_{L^\infty(\mathbb{T})} \leqslant 2^{1+1/2} \frac{m^{(1/2)^2} (\frac{1}{\alpha})^{1/2}}{|\frac{\lambda}{2} \log(\|f\|_{L^1(I)})|^{1/2(1+1/2)}}.$$

Proceeding thus repeatedly, we obtain for every $k \in \mathbb{N}^*$:

$$\|f\|_{L^\infty(\mathbb{T})} \leqslant 2^{b_k} \frac{m^{(1/2)^{k+1}} (\frac{1}{\alpha})^{c_k}}{|\frac{\lambda}{2} \log(\|f\|_{L^1(I)})|^{a_k}},$$

where $a_k$, $b_k$ and $c_k$ are three recurrent sequences satisfying:

$$a_1 = \frac{1}{2}\left(1 + \frac{1}{2}\right), \qquad b_1 = 1 + \frac{1}{2}, \qquad c_1 = \frac{1}{2}, \qquad a_{k+1} = \frac{1 + a_k}{2}, \qquad b_{k+1} = 1 + \frac{b_k}{2}, \qquad c_{k+1} = \frac{1 + c_k}{2}.$$

The proof of inequality (2) is thus completed by letting $k \to +\infty$.

To prove Eq. (3), we consider the sequence of functions:

$$u_n(z) = \frac{1}{(z - r)^n}, \quad \text{where } n \in \mathbb{N}^* \text{ and } 1 < r \leqslant n + 1.$$

Let $f_n = \frac{u_n}{\|u_n\|_{H^{1,\infty}}}$ be the $H^{1,\infty}$ normalized function of $u_n$. Then,

$$\|f_n\|_{L^\infty(\mathbb{T})} = \frac{r - 1}{n} \quad \text{and} \quad \|f_n\|_{L^\infty(I)} = \frac{(r - 1)^{n+1}}{n} \left(\frac{1}{1 + r^2}\right)^{n/2}.$$

Let $\varphi_n(r) = \|f_n\|_{L^\infty(\mathbb{T})} |\log(\|f_n\|_{L^\infty(I)})|$, then we have:

$$\varphi_n(r) = (r - 1)\frac{\log n}{n} + \frac{1}{2}(r - 1)\log\bigl(1 + r^2\bigr) - \frac{n + 1}{n}(r - 1)\log(r - 1).$$

Hence, for $r = \sqrt{n}$, it comes up that $\lim_{n \to \infty} \varphi_n(\sqrt{n}) = 1$, which completes the proof. □

Observe that the estimate (2) still holds for a simply connected bounded Jordan domain $\Omega$ in $\mathbb{R}^2$ with $\mathcal{C}^{1,\beta}$ boundary; $\beta \in ]0, 1[$. Note also that this estimate is false in the general setting where $f \in H^\infty$ only (we can consider the $H^\infty$ normalized function of $u_n$).

We deduce clearly from the second part of Theorem 2.4, that the estimate (2) is of optimal type: it is impossible to find a function $\varepsilon$ which tends to zero at zero such that for all $f \in \mathcal{B}_{1,\infty}$,

$$\|f\|_{L^\infty(\mathbb{T})} \leqslant \frac{1}{|\log(\|f\|_{L^1(I)})|} \varepsilon\bigl(\|f\|_{L^1(I)}\bigr).$$

The following corollary is a direct consequence of Theorem 2.4.

**Corollary 2.5.** *Let $K > 0$. There exists a non-negative constant $C_K$ such that any function $f \in H^{1,\infty}$ satisfying $\|f\|_{H^{1,\infty}} \leqslant K$ and $\|f\|_{L^1(I)} < 1$, also verifies the following inequality*:

$$\|f\|_{L^\infty(\mathbb{T})} \leqslant \frac{8/(1 - \frac{1}{2e}) \max(1, K)}{|\lambda \log(\|f\|_{L^1(I)})|}.$$

If we suppose that $f$ is sufficiently regular, we can improve inequality (2) in the same way as in the proof of Theorem 2.4.

**Theorem 2.6.** *Let $k \in \mathbb{N}^*$. There exists a non-negative constant $C_k$, such that for every $f \in \mathcal{B}_{k,\infty}$, we have*:

$$\|f\|_{L^\infty(\mathbb{T})} \leqslant \frac{C_k}{|\lambda \log(\|f\|_{L^1(I)})|^k}.$$

**Corollary 2.7.** *Let $K > 0$, $k$ and $m$ some integers with $0 \leqslant m < k$. Then there exists a non-negative constant $C = C(K, m, k)$, such that whenever $f \in H^{k,\infty}$ with $\|f\|_{H^{k,\infty}} \leqslant K$ and $\|f\|_{L^1(I)} < 1$, we have*:

$$\|f\|_{H^{m,\infty}} \leqslant \frac{C}{|\lambda \log(\|f\|_{L^1(I)})|^{k-m}}.$$

## 3. Applications

In this section, we consider two applications of the previous estimates. Let us consider first the Cauchy problem:

$$(CP) \quad \begin{cases} -\Delta u = 0 & \text{in } \mathbb{D}, \\ \partial_n u = \phi & \text{on } I, \\ u = f & \text{on } I, \end{cases}$$

where $\partial_n u$ stands for the partial derivative with respect to the outer normal derivative of $u$, $\Phi$ denotes the imposed current flux and $f$ the potential measurement. Then, we establish the following:

**Theorem 3.1.** *Let $\Phi \in \mathcal{C}^0(\bar{I})$, $c > 0$, $k$ and $m$ some integers with $0 \leqslant m < k$. We denote by $\mathcal{H}_c^k$ the set*:

$$\mathcal{H}_c^k = \left\{ v \in \mathcal{C}^k(\bar{\mathbb{D}}),\ \|v\|_{W^{k,\infty}(\bar{\mathbb{D}})} \leqslant c \right\}.$$

*Let $u_i \in \mathcal{H}_c^k$ the solution of $(CP)$ when $f = f_i$; $i = 1, 2$. If $\|f_1 - f_2\|_{L^1(I)} < 1$, then*

$$\|u_1 - u_2\|_{W^{m,\infty}(\mathbb{T})} \leqslant \frac{\beta}{|\log(\|f_1 - f_2\|_{L^1(I)})|^{k-m}},$$

*where $\beta > 0$ is a constant depending only upon $\Phi$, $I$, $m$, $k$ and $c$.*

We now look at the following inverse problem: Given a prescribed flux $\phi$ together with measurement $f$ on $I$, recover the function $q$ such that the solution $u$ to

$$(RP) \quad \begin{cases} -\Delta u = 0 & \text{in } \mathbb{D}, \\ \partial_n u = \phi & \text{on } I, \\ \partial_n u + qu = 0 & \text{on } \mathbb{T} \setminus I, \end{cases}$$

also satisfies $u_{|I} = f$.

Let $c, c' > 0$, $J = \mathbb{T} \setminus I$, and $K$ be a non-empty connected subset of $J$, for which the boundary does not intersect that of $I$. We suppose that $q$ belongs to the class of admissible Robin coefficients:

$$Q_{ad} = \left\{ q \in \mathcal{C}_0^2(\bar{J}),\ \left|q^{(k)}\right| \leqslant c',\ 0 \leqslant k \leqslant 2,\ \text{and } q \geqslant c\chi_K \right\}.$$

According to Theorem 2.4, we establish the following stability result:

**Theorem 3.2.** *Let $\phi \in W_0^{2,2}(I)$ be a positive function which is not identically trivial. There exists then, a non-negative constant $C$ such that for any $q_1, q_2 \in Q_{ad}$, we have*:

$$\|q_1 - q_2\|_{L^\infty(J)} \leqslant \frac{C}{|\log(\|u_1 - u_2\|_{L^1(I)})|},$$

*provided that $\|u_1 - u_2\|_{L^1(I)} < 1$, where $u_i$ denotes the solution of $(RP)$ with $q = q_i$; $i = 1, 2$.*

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