Available online at www.sciencedirect.com

ScienceDirect

C. R. Acad. Sci. Paris, Ser. I 347 (2009) 1217–1222

Comptes Rendus Mathematique

Numerical Analysis

A posteriori error estimate for a one-dimensional pollution problem in porous media ☆

Rajaa Aboulaich [a], Boujemâa Achchab [b], Aziz Darouichi [a]

[a] *LERMA, École Mohammadia d'ingénieurs, avenue Ibn-Sina B.P. 765, Agdal, Rabat, Maroc*
[b] *Université Hassan 1^er^, LM^{2}CE, FSEJ Settat et Ecole Supérieure de Technologie, B.P. 218 Berrechid, Maroc*

Received 18 March 2009; accepted after revision 9 July 2009

Available online 27 August 2009

Presented by Olivier Pironneau

Abstract

We are interested in the discretization of a time-dependent pollution problem modeling the mass transfer of contaminant in porous media, by the implicit Euler scheme in time and vertex-centered finite volumes in space. The error estimator consists of two types of computable error indicators, the first one being linked to the time discretization and the second one to the space discretization. ***To cite this article: R. Aboulaich et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Estimation a posteriori pour un problème de pollution unidimensionnel en milieu poreux. On s'intéresse à la discrétisation d'un problème de pollution instationnaire, modélisant le transfert de masse d'un polluant en milieu poreux, par un schéma d'Euler implicite en temps et par volumes finis centrés sur les nœuds en espace. L'estimateur d'erreur a posteriori développé ici est constitué de deux indicateurs d'erreur, le premier lié à la discrétisation temporelle, le second à la discrétisation spatiale. ***Pour citer cet article : R. Aboulaich et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Version française abrégée

Dans cette Note, nous nous intéressons à la discrétisation d'un problème parabolique intégro-différentiel (P) modélisant le transfert de masse d'un polluant en milieu poreux [2,14], par un schéma d'Euler implicite en temps et par un schéma volumes finis centré sur les nœuds en espace $(P)_h^n$. Nous proposons une estimation d'erreur a posteriori de type résidu. Celle-ci est composée de deux indicateurs, l'un lié à la discrétisation temporelle η_t^n (5), l'autre lié à la discrétisation spatiale η_{n,V_i} (12). Le terme intégral est approché par une formule de quadrature, supposée d'ordre assez élevée, de telle sorte que l'erreur d'intégration numérique soit négligée par rapport à l'erreur d'approximation.

☆ This work has been supported in part by the CNRST Morocco, Projet d'établissement Université Hassan 1^er^ Settat, the Volkswagen Foundation: Grant number I/79315, Hydro 3+3 project and Action Intégrée N^o^ MA/05, 115.

E-mail addresses: aboulaich@emi.ac.ma (R. Aboulaich), achchab@yahoo.fr (B. Achchab), azizdarouichi@gmail.com (A. Darouichi).

1631-073X/$ – see front matter © 2009 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.
doi:10.1016/j.crma.2009.07.017

Nous démontrons la borne supérieure globale et la borne inférieure locale pour chaque indicateur d'erreur (voir section 3).

1. Introduction

The finite volumes method (FVM) is a well-adapted method for the discretization of various types of partial differential equations and is extensively used in several engineering fields, see [10]. The mathematical analysis of the method has started only recently. In the context of finite element methods (FEM), an impressive amount of works has been done concerning the theory of a posteriori estimates and mesh adaptivity for a large class of equations; see, for instance [16]. Whereas, for FVM, the situation has not yet been as thoroughly explored, up to now only a few such results had been obtained, see for instance [5,12,19] and references therein. Plenty of work has been done concerning the a posteriori analysis of parabolic type problems. Part of it (see for instance [9,11]) deals only with time scheme adaptivity. However, the authors in [8] deal only with the space discretization and provide appropriate error indicators for it. Another concept consists in establishing a full time and space variational formulation of the continuous problem and using a discontinuous Galerkin method for the discretization with respect to all variables, see for instance [18]. In this Note, we follow a different approach, according to an idea of [7], which consists in introducing two different types of error indicators, both of them of residual type, one for the time discretization and one for the space discretization, and uncoupling as far as possible, the estimates of the time and space errors. It must be noted that both kinds of indicators only depend on the fully discrete solution and the data, so that computing them is easy and not expensive. In this paper we give a posteriori error estimates for the following integro-differential parabolic problem. For $T > 0$, and g, $f \in L^2(I)$, and $u_0 \in H^1(I)$: $u_0(1) = g(0)$ given functions, find u verifying:

$$(P)\quad \begin{cases} \partial_t u + Au = f + \nu \displaystyle\int_0^t u(s)\,\mathrm{d}s & \text{in } Q_T = (-1,1)\times(0,T),\\ \partial_x u(-1,t) = 0,\ u(1,t) = g(t) & \text{in } (0,T),\\ u(x,0) = u_0(x) & \text{in } I = (-1,1), \end{cases} \tag{1}$$

where $Au = -D\frac{\partial^2 u}{\partial x^2} + \frac{\partial(vu)}{\partial x} + (1-\nu)u$, $v \in C^2(\bar{I})$ is the transport velocity, $D > 0$ is the diffusion coefficient and ν is nonnegative constant. For the formulation of this problem see [4,6,14]. Let δ be the function: $\delta(x) = \frac{1}{D}\int_0^x v(s)\,\mathrm{d}s$. One sets, $L^2(I,\delta) = \{\phi : I \to \mathbb{R}$ measurable, $\int_I \phi^2(x)e^{-\delta(x)}\,\mathrm{d}x < +\infty\}$, $(\phi/\psi) = \int_I \phi(x)\psi(x)e^{-\delta(x)}\,\mathrm{d}x$, $\forall(\phi,\psi) \in (L^2(I,\delta))^2$ and $\|\phi\|^2_{L^2(I,\delta)} = (\phi/\phi)$ its associated norm. We need spaces $H^1_{\Gamma_D}(I) = \{\phi \in H^1(I); \phi(1) = 0\}$; $W(\text{resp. } W_0) = \{\phi \in L^2(0,T;H^1(I))$ (resp. $L^2(0,T;H^1_{\Gamma_D}(I))$); $\frac{\mathrm{d}}{\mathrm{d}t}\phi \in L^2(0,T;L^2(I))\}$. One defines the function θ of class $C^1(\bar{I})$ by: $\theta(x) = (1 - \nu + \frac{\mathrm{d}}{\mathrm{d}x}v(x))$, $\forall x \in \bar{I}$, and $a(\phi,\omega) = D(\frac{\mathrm{d}}{\mathrm{d}x}\phi/\frac{\mathrm{d}}{\mathrm{d}x}\omega) + (\theta\phi/\omega)$, $\forall(\phi,\omega) \in H^1(I)\times H^1(I)$.

Definition 1.1. A function $u \in W$ is a weak solution of problem (P) if $u - g \in W_0$; $u(0) = u_0$ in $L^2(I)$ and verifies for all $t \in (0,T)$ the variational equation:

$$\left(\frac{\partial}{\partial t}u(t)/\phi\right) + a\big(u(t),\phi\big) = (f/\phi) + \nu\left(\int_0^t u(s)\,\mathrm{d}s/\phi\right), \quad \forall\phi \in H^1_{\Gamma_D}(I).$$

The existence and uniqueness of solution of problem (P) is proved in [6]. We assume that exists a nonnegative constant $\beta > 0$ s.t. $\theta(x) \geqslant \beta$, $\forall x \in I$, which rules out all dominant advection problems.

2. A finite volume discretization

In the present section we shall discretize the considered problem (P) by implicit Euler scheme in time and vertex-centered finite volume method in space. In what follows, the symbol $\|.\|$ (resp. $\|.\|_V$, $V \subset I$) denotes the norm

$\|.\|_{L^2(I,\delta)}$ (resp. $\|.\|_{L^2(V,\delta)}$), and C a constant, possibly different at any occurrence, but independent of the mesh-size. Moreover, $a \preceq b$ means $a \leqslant Cb$. For each time t in $(0, T)$ and $\phi \in L^2(0,t; H^1_{\Gamma_D}(\Omega))$, we introduce the norm:

$$[\![\phi]\!](t) = \left(\|\phi(t)\|^2 + \int_0^t |\!|\!|\phi(s)|\!|\!|^2\,\mathrm{d}s\right)^{\frac{1}{2}}, \quad |\!|\!|\phi(s)|\!|\!|^2 = D\|\phi_x(s)\|^2 + \beta\|\phi(s)\|^2.$$

On choosing $\phi = u$ in Definition 1.1, integrating over $(0,t)$, and applying the Gronwall's lemma, lead to the following stability estimate. There is a positive constant c_0 s.t. $[\![u]\!](t) \leqslant c_0(\|u_0\|^2 + \frac{T}{\beta}\|f\|^2)^{\frac{1}{2}}$. Let $0 = t_0 < t_1 < \cdots < t_N = T$ be a partitioning of $[0,T]$ into variable time steps $\tau_n = t_n - t_{n-1}$, $n = 1,\ldots,N$. We set $\sigma_\tau = \max_{2\leqslant n\leqslant N}\frac{\tau_n}{\tau_{n-1}}$ and for each ϕ^n, we introduce the norm: $[\![\phi^n]\!] = (\|\phi^n\|^2 + \tau_n|\!|\!|\phi^n|\!|\!|^2)^{\frac{1}{2}}$. Likewise, we have the following stability estimate

$$[\![u^n]\!] \leqslant \left(\|u^{n-1}\|^2 + \frac{\tau_n}{\beta}\|f\|^2 + \tau_n c_1\right)^{\frac{1}{2}}, \tag{2}$$

where $c_1 = 2LKTe^{LT}$, $K = \|u_0\|^2 + \frac{T}{\beta}\|f\|^2$ and $L = \nu T$. For each n, $0 \leqslant n \leqslant N$, let $(\mathcal{T}_h^n)_h$ be a partition of I into subintervals. $\mathcal{V}_h^n$ is chosen as the set of M control volumes (or cells) V_i that constitute the dual of the mesh $(\mathcal{T}_h^n)$, and such that $\bar{I} = \bigcup_{i=1,\ldots,M} V_i$. Let $\mathcal{Z}_h$ be the set of all the nodes of the $(\mathcal{T}_h^n)_h$ and let $\mathcal{Z}_h^\circ$ be the subset of the internal nodes, that is, $\mathcal{Z}_h^\circ = \mathcal{Z}_h \setminus \{-1,1\}$, and let $\mathcal{Z}_{h,N}$ be the subset of the related nodes to the Neumann condition, that is, $\mathcal{Z}_{h,N} = \mathcal{Z}_h \setminus \{\mathcal{Z}_h^\circ \cup \{1\}\}$. We denote by Γ_h^n the set of nodes γ of the dual decomposition $\mathcal{V}_h^n$. We may construct another partition of I, denoted by $\mathcal{Q}_h^n$ and formed by intervals Q defined by $Q = V_i \cap T$. We also need to define the set $\mathcal{K}_h^i = \bigcup K$, formed by the intervals K having x_{V_i} (the center of the control volume V_i) as a bound and γ as the second one. Next we define the spaces $\mathcal{P}_1(\mathcal{T}_h^n) := \{v_h \in C^0(\bar{I})\colon\ v_h|T \in \mathcal{P}_1;\ \forall T \in \mathcal{T}_h^n\}$, and $\mathcal{P}_0(\mathcal{V}_h^n) := \{w_h \in L^2(\bar{I})\colon\ w_h|V_i \in \mathcal{P}_0;\ i = 1,\ldots,M\}$, where $\mathcal{P}_l$ is the set of polynomial functions of degree $\leqslant l$. We denote by h_T (resp. h_i) the length of $T \in \mathcal{T}_h^n$ (resp. $V_i \in \mathcal{V}_h^n$) and, for a node z which is shared between T_1 and T_2, h_z is the minimum of h_{T_1} and h_{T_2}. Setting $V_{h,0} = V_h \cap H_{\Gamma_D}(I) = \{v_h \in \mathcal{P}_1(\mathcal{T}_h^n)$ and $v_h(1) = 0\}$, and denoting by u_h^n the approximation of the exact solution u^n in $V_{h,0}$, namely $u_h^n = \sum_{i=1}^{M-1} u_i^n\psi_i(x)$ with ψ_i is the shape function associated with node x_i. Let us now consider the following finite volumes approximation of (P):

$$(P)_h^n \quad \begin{cases} \text{Find } (u_h^n - g^n)_{0\leqslant n\leqslant N} \in (V_{h,0})^{N+1} \text{ such that: } \quad u_h^0 = \Pi_h u_0 \quad \text{in } \Omega, \quad \text{and} \\ \displaystyle\int_{V_i} \frac{u_h^n - u_h^{n-1}}{\tau_n} + D\big[F^n_{h,i+\frac{1}{2}} - F^n_{h,i-\frac{1}{2}}\big] + \int_{V_i} v_h\frac{\partial u_h^n}{\partial x}\,\mathrm{d}x + \int_{V_i}\theta_h u_h^n\,\mathrm{d}x = \int_{V_i} f\,\mathrm{d}x + \nu\int_{V_i}\mathcal{I}u_h^n(x)\,\mathrm{d}x \\ \text{for all } i = 1,\ldots,M,\ n = 1,\ldots,N, \end{cases}$$

where Π_h is the L^2-projection on $V_{h,0}$.

The numerical diffusion flux function $F^n_{h,i+\frac{1}{2}}$ is chosen such that we have the local conservativity. In order to calculate this numerical diffusion flux, we define the primal partition of I by: $(\mathcal{T}_h^n)_{h>0} : \{x_i\}_{i=1,\ldots,M}$ with $I_i = [x_i, x_{i+1}]$ $\forall i = 1,\ldots,M-1$ and $h_{i+\frac{1}{2}} = x_{i+1} - x_i$, and the corresponded dual partition by: $(\mathcal{V}_h^n)_{h>0}$ with $V_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$ $\forall i = 2,\ldots,M-1$, $V_1 = [x_1, x_{\frac{3}{2}}]$ with $x_1 = x_{\frac{1}{2}}$ and $V_M = [x_{M-\frac{1}{2}}, x_M]$ with $x_M = x_{M+\frac{1}{2}}$. Therefore, $F^n_{h,i+\frac{1}{2}} = \frac{\partial u_h^n}{\partial x}(x_{i+\frac{1}{2}})$, which yields $F^n_{h,i+\frac{1}{2}} = \frac{u^n_{i+1} - u^n_i}{h_{i+\frac{1}{2}}}$. The approximation of v (resp. θ), v_h (resp. θ_h), is a piecewise polynomial of degree smaller than a fixed integer l and such that there exists a constant $c(v)$ (resp. $c(\theta)$) only depending on v (resp. θ) satisfying

$$\|v - v_h\|_{L^\infty(I)} \leqslant c(v)h^{l+1} \quad \text{and} \quad \|\theta - \theta_h\|_{L^\infty(I)} \leqslant c(\theta)h^{l+1}. \tag{3}$$

The quantity $\mathcal{I}u_h^n$ is a numerical approximation of the integral $\mathcal{I}u^n$ and there exist two constants $c(u)$ depending on u and $k > 1$ a fixed constant such that

$$\big\|\mathcal{I}u_h^n - \mathcal{I}u^n\big\|_V \leqslant c(u)\tau^k h_V. \tag{4}$$

Remark 1. The terms $\int_{V_i} \frac{u_h^n - u_h^{n-1}}{\tau_n}\,\mathrm{d}x$, $\int_{V_i} v_h \frac{\partial u_h^n}{\partial x}\,\mathrm{d}x$ and $\int_{V_i} \theta_h u_h^n\,\mathrm{d}x$ can be approximated by $h_i \frac{u_i^n - u_i^{n-1}}{\tau_n}$, $v_i[u_h^n(x_{i+\frac{1}{2}}) - u_h^n(x_{i-\frac{1}{2}})] = v_i[u_i^n - u_{i-1}^n]$ for upwind scheme and $h_i\theta_i u_i^n$ respectively where $u_i^n = u_h^n(x_i)$, $v_i = v_h(x_i)$ and $\theta_i = \theta_h(x_i)$, the existence and uniqueness of a solution of this scheme is proved in [1].

3. A posteriori error analysis

3.1. A posteriori error analysis of the time discretization

By analogy to [7,11,13], for each n, $1 \leqslant n \leqslant N$, we define the time error indicators

$$\eta_t^n = \left(\frac{\tau_n}{3}\right)^{\frac{1}{2}} \left(D^{\frac{1}{2}} \left\|\partial_x\left(u_h^n - u_h^{n-1}\right)\right\| + \beta^{\frac{1}{2}} \left\|u_h^n - u_h^{n-1}\right\|\right). \tag{5}$$

It can be observed that, once the discrete solution $(u_h^n)_{0\leqslant n\leqslant N}$ is known, the previous error indicators are very easy to compute.

Proposition 3.1. *The following time upper error estimate holds, for* $1 \leqslant n \leqslant N$,

$$[\![u - u_\tau]\!](t_n) \preceq \left(\left(1 + \sigma_\tau^{\frac{1}{2}}\right)[\![u_\tau - u_{h\tau}]\!](t_n) + \left(\sum_{m=1}^{n} \left(\eta_t^m\right)^2\right)^{\frac{1}{2}} + c_n(u_0, f)\left(\max_{1\leqslant m\leqslant n} \tau_m\right)\right). \tag{6}$$

Moreover, we have the following time lower error bound

$$\begin{aligned}\eta_t^n \preceq \Bigg(&\left\|\frac{\partial(u - u_\tau)}{\partial t}\right\|_{L^2(t_{n-1},t_n;L^2(I,\delta))} + D^{\frac{1}{2}} \left\|\partial_x(u - u_\tau)\right\|_{L^2(t_{n-1},t_n;L^2(I,\delta))} + \|u - u_\tau\|_{L^2(t_{n-1},t_n;L^2(I,\delta))} \\ &+ c_n(u_0, f)\tau_n + [\![u^n - u_h^n]\!] + \sigma_\tau^{\frac{1}{2}} [\![u^{n-1} - u_h^{n-1}]\!]\Bigg),\end{aligned} \tag{7}$$

where the constant $c_n(u_0, f)$ *is given by* $c_n(u_0, f) = \frac{\nu}{\sqrt{3}} \beta^{-\frac{1}{2}} t_n^{\frac{1}{2}} K^{\frac{1}{2}} e^{\frac{LT}{2}}$.

Proof. For $\phi \in H^1_{\Gamma_D}(I)$, we have

$$\left(\frac{\partial(u - u_\tau)}{\partial t}/\phi\right) + a(u - u_\tau, \phi) = -a\left(u_\tau - u^n, \phi\right) + \nu\left(\int_{t_n}^{t} u(s)\,\mathrm{d}s/\phi\right), \tag{8}$$

taking $\phi = u - u_\tau$, integrating over $[t_{n-1}, t_n]$, sum up on the n, and applying the Hölder's inequality, and owing to the definitions of u_τ, σ_τ and η_t^m we can get the time upper error bound. By virtue of (5), taking $\phi = u^n - u^{n-1}$ in (8), integrating over (t_{n-1}, t_n), and applying the Hölder's inequality, this yields the time lower error bound. □

3.2. A posteriori error analysis of the space discretization

We now introduce the usual nodal basis $\{\phi_z\}_{z\in\mathcal{Z}_h}$ for V_h, whence $\{\phi_z\}_{z\in\mathcal{Z}_h^\circ}$ is the nodal basis for $V_{h,0}$, and the following quasi-interpolants given in [15], defined for a given $\psi \in H^1(I)$ by: $I_h\psi \in V_h$ as $I_h\psi := \sum_{z\in\mathcal{Z}_h} \frac{\int_\Omega \phi_z\psi}{\int_\Omega \phi_z}\phi_z$, and $I_{h,0}\psi \in V_{h,0}$ as $I_{h,0}\psi := \sum_{z\in\mathcal{Z}_h^\circ} \frac{\int_\Omega \phi_z\psi}{\int_\Omega \phi_z}\phi_z$. Let $T \in \mathcal{T}_h^n$ and let ω_T be the union of T and the one or two adjacent elements. Let $z \in \mathcal{Z}_h$ and let V be a control volume which z belongs to, using [17, Lemma 3.1], we derive

$$\sum_{z\in\mathcal{Z}_h^\circ\cap V} h_z^{-1}(I_{h,0}\psi)^2(z) \preceq |\psi|^2_{1,\omega_V}. \tag{9}$$

Lemma 3.2. *Let ψ be an element of $H^1_{\Gamma_D}(I)$ and $\psi_h = I_{h,0}\psi$, and let $\overline{\psi_h}$ be the mean value of ψ_h over the control volume V, then one has the following estimates*:

$$\sum_{z\in\mathcal{Z}_h^\circ\cap V} h_z^{-1}(\psi_h - \overline{\psi_h})^2(z) \preceq |\psi|^2_{1,\omega_V}, \quad \|\psi_h - \overline{\psi_h}\|_{0,V} \preceq h_V|\psi|_{1,\omega_V}, \tag{10}$$

and

$$\|\psi_h - \overline{\psi_h}\|_{0,V} \preceq h_V|\psi|_{1,\omega_V}. \tag{11}$$

In order to obtain an a posteriori estimate in each control volume V_i, let us define the following local spatial indicators

$$\eta^2_{n,V_i} := h_i^2\|R_h^n\|^2_{V_i} + \sum_{z\in\mathcal{Z}_h^\circ\cap V_i} h_z\left(D\left[\frac{\partial u_h^n}{\partial x}\right](z)\right)^2 + \sum_{z\in\mathcal{Z}_{h,N}\cap V_i} h_z\left(D\left|\frac{\partial u_h^n}{\partial x}(z)\right|\right)^2, \tag{12}$$

where $R_h^n = f_h + \nu\mathcal{I}u_h^n - \frac{u_h^n - u_h^{n-1}}{\tau_n} + D\frac{\partial^2 u_h^n}{\partial x^2} - v_h\frac{\partial u_h^n}{\partial x} - \theta_h u_h^n$. We have the following upper bounds for the error:

Proposition 3.3. *If u is the solution of problem* (1) *and $u_{h\tau}$ is a continuous approximation, linear on time and on space defined from the value $(u_h^n)_n$, solution of problem $(P)_h^n$, then we have the following estimate*:

$$\begin{aligned}[\![u_\tau - u_{h\tau}]\!](t_n) \preceq{}& \left(\sum_{m=1}^{n}\tau_m\sum_{i=1}^{M}\left(\eta^2_{m,V_i} + h_i^2\|f - f_h\|^2_{V_i}\right)\right)^{\frac12} + \|u_0 - \Pi_h u_0\| + c_n^*(u_0,f)h^{(l+1)}\\ &+ c_n(u)h\tau^{k+1} + h^2\tau\mathcal{B}_n\left(u_h^n\right),\end{aligned} \tag{13}$$

where $c_n^*(u_0,f) = \|u_0\| + t_n^{\frac12}\|f\| + t_n^{\frac12}c_0^{\frac12} + \tau_1^{\frac12}|\!|\!|u_0|\!|\!|$, $c_n(u) = c(u)t_n^{\frac12}$, and

$$\mathcal{B}_n\left(u_h^n\right) = \left(\sum_{m=1}^{n}\sum_{i=1}^{M}\left\|D\frac{\partial u_h^m}{\partial x}\right\|^2_{0,\partial V_i} + \left\|\frac{\partial u_h^m}{\partial x}\right\|^2_{0,I}\right)^{\frac12}.$$

We have also the following lower bounds for the error:

Proposition 3.4. *For $f\in L^2(I)$ and $u_0\in H^1(I)$, we have the following local estimate*:

$$\begin{aligned}\eta_{n,V_i} \preceq{}& h_i\left\|\frac{\partial(u_\tau - u_{h\tau})}{\partial t}\right\|_{0,V_i} + \left\|D^{\frac12}\partial_x\left(u^n - u_h^n\right)\right\|_{0,V_i} + h_i\|f - f_h\|_{0,V_i} + h_i\left\|\theta^{\frac12}\left(u^n - u_h^n\right)\right\|_{0,V_i}\\ &+ h^{l+1}c_n\left(u_h^n\right) + h_i^2\tau^k c(u),\end{aligned} \tag{14}$$

where $c_n(u_h^n) = c(v)\|\frac{\partial u_h^n}{\partial x}\|_{0,V_i} + c(\theta)\|u_h^n\|_{0,V_i}$.

For the proofs of the Propositions 3.2 and 3.3, we can see [3].

Remark 2. Some constants $c(u)$ appearing in the upper and lower bounds depending on the exact solution u, but pondering with coefficient of higher order. This is due to the numerical integration order.

We do not actually show an a posteriori estimator for the multidimensional case or the advection dominated case ($\beta < 0$); this is of interest but lies outside the scope of this Note.

References

[1] R. Aboulaich, B. Achchab, A. Darouichi, Approximation par volumes finis d'un problème de pollution en milieu poreux, in: Proceeding of 8th Days of Numerical Analysis and Optimization, ENIM-Rabat, Morocco, 2005.

[2] R. Aboulaich, B. Achchab, A. Darouichi, D. Meskine, A study of a pollution problem in porous media during vapor extraction, J. Math. Sci.: Adv. Appl. 1 (3) (2008) 635–662.
[3] R. Aboulaich, B. Achchab, A. Darouichi, A posteriori error analysis in finite volumes approximations for a pollution problem in porous media, Internal report, LERMA, EMI Rabat, 2009.
[4] R. Aboulaich, S. Afilal, J. Pousin, Sur un modèle de pollution en mileux poreux, Maths recherche et application 2 (2000) 108–120.
[5] B. Achchab, A. Agouzal, G.C. Buscaglia, F. Oudin, Adaptive finite volume scheme for elliptic problems, submitted for publication.
[6] S. Afilal, Sur un problème d'identification en milieu poreux, Thèse d'Etat, Ecole Mohammadia d'Ingénieurs Rabat, 7 mars 2000.
[7] A. Bergam, C. Bernardi, Z. Mghazli, A posteriori analysis of the finite element discretization of some parabolic equations, Math. Comp. 74 (251) (2004) 1117–1138.
[8] C. Bernardi, B. Métivet, Indicateurs d'erreur pour l'équation de la chaleur, Rev. Européenne Élém. Finis 9 (2000) 425–438.
[9] M. Bieterman, I. Babuska, The finite element method for parabolic problems. II. A posteriori error estimation and adaptive approach, Numer. Math. 40 (1982) 339–371.
[10] R. Eymard, T. Gallouet, R. Herbin, Finite volumes method, in: P.G. Ciarlet, J.L. Lions (Eds.), Handbook of Numerical Analysis, vol. 7, September 1997.
[11] C. Johnson, Y.-Y. Nie, V. Thomée, An a posteriori error estimate and adaptive timestep control for a backward Euler discretization of parabolic problem, SIAM J. Numer. Anal. 27 (1990) 277–291.
[12] S. Nicaise, A posteriori error estimations of some cell centered finite volume methods for diffusion–convection–reaction problems, SIAM J. Numer. Anal. 44 (2006) 949–978.
[13] M. Picasso, Adaptive finite elements for a linear parabolic problem, Comput. Methods Appl. Mech. Engrg. 167 (1998) 223–237.
[14] J. Pousin, A. Roukbi, R. Gourdon, P. Le Goff, P. Moszkowicz, Évaporation d'une substance organique dans un milieu poreux, C. R. Acad. Sci. Paris, Sér. II b 327 (1999) 371–377.
[15] G. Sangalli, Robust a posteriori estimator for advection–diffusion–reaction problems, M2AN 39 (2) (2005) 319–348.
[16] R. Verfürth, A Review of a Posteriori Error Estimation and Adaptive Mesh-Refinement Techniques, Wiley & Teubner, 1996.
[17] R. Verfürth, A posteriori error estimators for convection–diffusion equations, Numer. Math. 4 (1998) 641–663.
[18] R. Verfürth, A posteriori error estimation techniques for nonlinear elliptic and parabolic PDES, Rev. Européenne Élém. Finis. 9 (2000) 377–402.
[19] M. Vohralìk, Residual flux-based a posteriori error estimates for finite volume and related locally conservative methods, Numer. Math. 111 (1) (2008) 121–158.