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Cartan homotopy formulae and the bivariant Hochschild complex

Abhishek Banerjee

Department of Mathematics, Johns Hopkins University, 3400 N Charles St., 404 Krieger Hall, Baltimore, MD 21218, USA

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Abstract

The formula $L_D = [e_D + E_D, \bar{b} + \overline{B}]$ (see Goodwillie [Cyclic homology, derivations, and the free loopspace, Topology 24 (2) (1985) 187–215]) on the normalized Hochschild complex is the standard replacement in noncommutative geometry for the classical Cartan homotopy formula. Our purpose is to extend this formula to the normalized bivariant Hochschild complex. ***To cite this article: A. Banerjee, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Formule homotopique de Cartan et le complexe de Hochschild bivariant. La formule $L_D = [e_D + E_D, \bar{b} + \overline{B}]$ (voir Goodwillie [Cyclic homology, derivations, and the free loopspace, Topology 24 (2) (1985) 187–215]) sur le complexe de Hochschild normalisé joue le rôle, en géométrie non commutative, de la formule homotopique de Cartan en homologie de Rham. Notre but est détendre cette formule au complexe de Hochschild bivariant normalisée. ***Pour citer cet article : A. Banerjee, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Cyclic homology, introduced by Connes [1,2] and Tsygan [7], acts as a noncommutative analogue of de Rham cohomology. If A is an algebra over a commutative ring k and D is a k-linear derivation on A, then the derivation induces maps L_D, e_D and E_D on the normalized Hochschild complex $\overline{C}^h_*(A)$ of A, satisfying the formula:

$$L_D = [e_D + E_D, \bar{b} + \overline{B}], \tag{1}$$

where $\bar{b}$ is the differential on the normalized Hochschild complex and $\overline{B}$ is the normalized Connes differential (see [5, §2.1.9]). In noncommutative geometry, (1) plays the role of Cartan homotopy formula. Formula (1) is due to Goodwillie [3] (see also the paper of Rinehart [6]). The purpose of this Note is to extend this formula to the normalized bivariant Hochschild complex.

The formulation of bivariant Hochschild cohomology (and bivariant cyclic cohomology) that we use is the one due to Jones and Kassel [4]. In particular, the bivariant cyclic cohomology due to [4] unifies negative cyclic homology and cyclic cohomology and also bears formal similarities to Kasparov's KK-theory.

E-mail address: abanerje@math.jhu.edu.

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Throughout, we shall use the notations and terminology of [5]. Let A denote a unital algebra over a commutative ground ring k and let D be a k-linear derivation on A. The bivariant Hochschild cohomology group of A will be denoted by $HH^*(A,A)$. We set $C_n(A)=A^{\otimes n+1}$, the tensor products being taken over k. For sake of convenience, we shall often denote the element $a_0\otimes a_1\otimes\cdots\otimes a_n$ of $A^{\otimes n+1}$ by $(a_0,a_1,\ldots,a_n)$.

Let $C_*^h(A)$ denote the complex $(C_*(A),b)$, where b is the Hochschild differential (see [5, Chapter 1] for a description of b). Let B denote Connes' map $B:C_*(A)\to C_{*+1}(A)$ (see [5, Chapter 2] for a description of this map). Setting $\overline{A}=A/k$ and $\overline{C}_n(A)=A\otimes\overline{A}^{\otimes n}$ gives us a normalized Hochschild complex $(\overline{C}_*^h(A),\bar{b})$ that is quasi-isomorphic to the Hochschild complex $(C_*^h(A),b)$. The map B also descends to a map $\overline{B}:\overline{C}_*(A)\to\overline{C}_{*+1}(A)$. We shall denote $\bar{b}$ and $\overline{B}$ simply by b and B respectively. Also, throughout this paper, all commutators $[.,.]$ are understood to be in the graded sense.

1. Bivariant cyclic cohomology and derivations

We briefly recall the definition of bivariant Hochschild cohomology. For details, see [5, Section 5.1], or the original paper of Jones and Kassel [4].

Definition 1.1. (See [5, §5.5.1-2].) Consider the graded module $\mathrm{Hom}(\overline{C}_*^h(A),\overline{C}_*^h(A))$ which in degree p is defined to be

$$\mathrm{Hom}^p\big(\overline{C}_*^h(A),\overline{C}_*^h(A)\big):=\prod_n\mathrm{Hom}\big(\overline{C}_n(A),\overline{C}_{n+p}(A)\big).\tag{2}$$

Then the differential ∂_h on $\mathrm{Hom}(\overline{C}_*^h(A),\overline{C}_*^h(A))$ is defined by $\partial_h(f)=bf-(-1)^{|f|}fb$, f being a homogeneous element of $\mathrm{Hom}(\overline{C}_*^h(A),\overline{C}_*^h(A))$ of degree $|f|$ and b being the Hochschild differential. The homology of this complex is referred to as the bivariant Hochschild cohomology of A and is denoted by $HH^*(A,A)$.

There is another operator ∂_c on the module $\mathrm{Hom}(\overline{C}_*^h(A),\overline{C}_*^h(A))$ defined as $\partial_c(f)=(b+B)f-(-1)^{|f|}f(b+B)$ for a homogeneous element f of degree $|f|$. Now, we prove the following lemma:

Lemma 1.2. *Let $M_p=\{M_p^n\}_{n\in\mathbb{Z}}$ denote a sequence of maps $M_p^n:\overline{C}_n(A)\to\overline{C}_{n+p}(A)$, $n\geqslant 0$ (we do not assume that the maps M_p^n commute with either B or b). Given $f\in\mathrm{Hom}(\overline{C}_*^h(A),\overline{C}_*^h(A))$ of homogeneous degree q, define*

$$M_p^{n,h}(f):\overline{C}_n(A)\to\overline{C}_{n+p+q}(A),\qquad M_p^{n,h}(f):=M_p^{q+n}f-(-1)^{pq}fM_p^n.\tag{3}$$

The collection of elements $M_p^{n,h}(f):\overline{C}_n(A)\to\overline{C}_{n+p+q}(A)$, $n\geqslant 0$ defines a homogeneous element of $\mathrm{Hom}(\overline{C}_*^h(A),\overline{C}_*^h(A))$ of degree $q+p$. This defines a morphism $M_p^h:\mathrm{Hom}(\overline{C}_*^h(A),\overline{C}_*^h(A))\to\mathrm{Hom}(\overline{C}_*^h(A),\overline{C}_*^h(A))$ of degree p. Then, on the module $\mathrm{Hom}(\overline{C}_*^h(A),\overline{C}_*^h(A))$, we have:

$$\begin{aligned}&\big[\partial_h,M_p^h\big](f)=[b,M_p]f+(-1)^{pq+p+q}f[M_p,b],\\&\big[\partial_c,M_p^h\big](f)=[b+B,M_p]f+(-1)^{pq+p+q}f[M_p,b+B].\end{aligned}\tag{4}$$

Proof. We calculate, for $f\in\mathrm{Hom}^q(\overline{C}_*^h(A),\overline{C}_*^h(A))$;

$$\begin{aligned}\partial_hM_p^h(f)&=bM_p^h(f)-(-1)^{p+q}M_p^h(f)b\\&=\big(bM_pf-(-1)^{pq}bfM_p\big)-\big((-1)^{p+q}M_pfb-(-1)^{pq+p+q}fM_pb\big),\\M_p^h\partial_h(f)&=M_p\partial_h(f)-(-1)^{p(q-1)}\partial_h(f)M_p\\&=\big(M_pbf-(-1)^qM_pfb\big)-(-1)^{p(q-1)}\big(bfM_p-(-1)^qfbM_p\big),\\\big[\partial_h,M_p^h\big](f)&=\partial_hM_p^h(f)-(-1)^pM_p^h\partial_h(f)=[b,M_p]f+(-1)^{pq+p+q}f[M_p,b].\end{aligned}\tag{5}$$

If $f\in\mathrm{Hom}^q(\overline{C}_*^h(A),\overline{C}_*^h(A))$, $\partial_c(f)$ is not a homogeneous element of $\mathrm{Hom}(\overline{C}_*^h(A),\overline{C}_*^h(A))$. However, since both operators b and B are of odd degree, this fact does not interfere with the powers of (-1) appearing in (5). Hence, we can show that $[\partial_c,M_p^h](f)=[b+B,M_p]f+(-1)^{pq+p+q}f[M_p,b+B]$. □

Proposition 1.3. (a) *Let D be a derivation on A. Extend D to $C_n(A) = A^{\otimes n+1}$, $n \geqslant 0$, by setting $L_D(a_0, a_1, \ldots, a_n) = \sum_{i=0}^{n}(a_0, \ldots, Da_i, \ldots, a_n)$. Then L_D induces a morphism $L_D^h : HH^n(A, A) \to HH^n(A, A)$.*

(b) *Let u be an element of A and consider the inner derivation $D(a) = [u, a]$. Then the morphism L_D^h: $HH^n(A, A) \to HH^n(A, A)$ is zero.*

Proof. (a) Clearly, L_D descends to an operator of degree 0 on $\overline{C}_*^h(A)$; so we can take $p = 0$ in Lemma 1.2 and define $L_D^h(f) = L_D f - f L_D$ for a homogeneous element f. We can check that $[b, L_D] = 0$, from which it follows that $\partial_h L_D^h - L_D^h \partial_h = [\partial_h, L_D^h] = 0$ and hence L_D induces a morphism L_D^h on bivariant Hochschild cohomology.

(b) Let $f \in \mathrm{Hom}^q(\overline{C}_*^h(A), \overline{C}_*^h(A))$ be such that $\partial_h(f) = 0$. Then $bf = (-1)^q fb$. If we let $h(u) : \overline{C}_n(A) \to \overline{C}_{n+1}(A)$ be the map $h(u)(a_0, a_1, \ldots, a_n) = \sum_{0 \leqslant i \leqslant n}(-1)^i(a_0, \ldots, a_i, u, a_{i+1}, \ldots, a_n)$, then $bh(u) + h(u)b = -L_D$. Then, using the notation of Lemma 1.2 and setting $bf = (-1)^q fb$, we have:

$$-L_D^h(f) = -L_D f + f L_D = \big(bh(u) + h(u)b\big)f - f\big(bh(u) + h(u)b\big) = \partial_h\big(h(u)f\big) - (-1)^q \partial_h\big(fh(u)\big)$$

and hence $L_D^h(\operatorname{Ker} \partial_h) \subseteq \operatorname{Im}(\partial_h)$. $\square$

2. Cartan homotopy formulae

In this section, we shall prove the extension of Cartan homotopy formula to the bivariant context. Given a k-linear derivation D on A, there are the following two additional operators on $\overline{C}_*^h(A)$; $e_D : \overline{C}_n(A) \to \overline{C}_{n-1}(A)$ and $E_D : \overline{C}_n(A) \to \overline{C}_{n+1}(A)$, which, along with the operator L_D, satisfy (1), i.e., the noncommutative analogue of Cartan homotopy formula:

$$\begin{aligned} e_D(a_0, \ldots, a_n) &:= (-1)^{n+1}\big(D(a_n)a_0, a_1, \ldots, a_{n-1}\big), \\ E_D(a_0, \ldots, a_n) &:= \sum_{1 \leqslant i \leqslant j \leqslant n} (-1)^{in+1}\big(1, a_i, a_{i+1}, \ldots, a_{j-1}, D(a_j), a_{j+1}, \ldots, a_n, a_0, \ldots, a_{i-1}\big). \end{aligned} \tag{6}$$

Then, in the notation of Lemma 1.2, we can use the operators L_D, e_D and E_D on $\overline{C}_*^h(A)$ to define operators L_D^h, e_D^h and E_D^h on $\mathrm{Hom}(\overline{C}_*^h(A), \overline{C}_*^h(A))$.

Proposition 2.1. *On the module* $\mathrm{Hom}(\overline{C}_*^h(A), \overline{C}_*^h(A))$, *the operators e_D^h, E_D^h and L_D^h satisfy*:

$$(1) \quad \big[\partial_h, e_D^h\big] = 0, \qquad (2) \quad \big[\partial_c, e_D^h\big] + \big[\partial_c, E_D^h\big] = L_D^h.$$

Proof. Taking $p = -1$ in Lemma 1.2, we know that, if f is a homogeneous element of degree q in $\mathrm{Hom}(\overline{C}_*^h(A), \overline{C}_*^h(A))$, we have $[\partial_h, e_D^h](f) = [b, e_D]f - f[e_D, b]$. Since $[e_D, b] = 0$ on the normalized complex $\overline{C}_*^h(A)$, (1) follows. Also,

$$\begin{aligned} \big[\partial_c, e_D^h\big](f) &= [b, e_D]f - f[e_D, b] + [B, e_D]f - f[e_D, B] = [B, e_D]f - f[e_D, B], \\ \big[\partial_c, E_D^h\big](f) &= [b, E_D]f - f[E_D, b] + [B, E_D]f - f[E_D, B] = [b, E_D]f - f[E_D, b]. \end{aligned}$$

From the analogue of the Cartan homotopy formula in Hochschild cohomology, we have $[e_D, B] + [E_D, b] = L_D$ and hence (note that $[e_D, B] = [B, e_D]$ and $[E_D, b] = [b, E_D]$) $([\partial_c, e_D^h] + [\partial_c, E_D^h])(f) = ([B, e_D] + [b, E_D])f - f([e_D, B] + [E_D, B]) = L_D f - f L_D = L_D^h(f)$. $\square$

Given two derivations D and D' on A, we know, from [6], that $[L_D, e_{D'}] = e_{[D, D']}$. We now prove the analogue of this result.

Proposition 2.2. *Let D and D' be two derivations on A. Then, on the module* $\mathrm{Hom}(\overline{C}_*^h(A), \overline{C}_*^h(A))$, *we have the formula* $[L_D^h, e_{D'}^h] = e_{[D, D']}^h$.

Proof. Let f be an element of $\mathrm{Hom}(\overline{C}_*^h(A), \overline{C}_*^h(A))$ of degree q. Then:

$$
\begin{aligned}
L_D^h e_{D'}^h(f) &= L_D^h\big(e_{D'}f-(-1)^q f e_{D'}\big) = \big(L_D e_{D'}f-(-1)^q L_D f e_{D'}\big)-\big(e_{D'}fL_D-(-1)^q f e_{D'}L_D\big),\\
e_{D'}^h L_D^h(f) &= e_{D'}^h(L_D f - fL_D) = (e_{D'}L_D f - e_{D'}fL_D)-(-1)^q(L_D f e_{D'} - fL_D e_{D'}),\\
\big[L_D^h, e_{D'}^h\big](f) &= L_D e_{D'}(f) - e_{D'}L_D(f) = [L_D, e_{D'}](f)-(-1)^q f[L_D, e_{D'}]\\
&= e_{[D,D']}f-(-1)^q f e_{[D,D']} = e_{[D,D']}^h(f). \qquad \square
\end{aligned}
$$

Finally, let $\mathrm{Der}_k(A)$ denote the module of k-linear derivations on A and let $\mathrm{Inn}(A)$ denote the submodule of inner derivations. Then, recall that; for the first Hochschild cohomology group $H^1(A,A)$ of A with coefficients in A, we know that $H^1(A,A)\cong \mathrm{Der}_k(A)/\mathrm{Inn}(A)$ and that the standard commutator on derivations makes $H^1(A,A)$ into a Lie algebra. Therefore, we have:

Proposition 2.3. *For any $n\geqslant 0$, the module $HH^n(A,A)$ carries a Lie algebra action of $H^1(A,A)$.*

Proof. Let $D\in \mathrm{Der}_k(A)$. From Proposition 1.3(a), we know that D induces a morphism $L_D^h : HH^n(A,A)\to HH^n(A,A)$, $\forall n\geqslant 0$. From Proposition 1.3(b), we know that if D is an inner derivation, then $L_D^h=0$. Hence, each element $x\in H^1(A,A)\cong \mathrm{Der}_k(A)/\mathrm{Inn}(A)$ induces an operator $L_x^h : HH^n(A,A)\to HH^n(A,A)$. Furthermore, it is easy to check that given two derivations D and D' on A, $[L_D^h, L_{D'}^h]=L_{[D,D']}^h$, whence it follows that $H^1(A,A)$ has a Lie algebra action on each $HH^n(A,A)$. $\square$

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