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Lie Algebras/Harmonic Analysis

Generalized Fourier transforms $\mathcal{F}_{k,a}$

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Abstract

We construct a two-parameter family of actions $\omega_{k,a}$ of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ by differential-difference operators on $\mathbb{R}^N$. Here, k is a multiplicity-function for the Dunkl operators, and $a > 0$ arises from the interpolation of the Weil representation and the minimal unitary representation of the conformal group. The action $\omega_{k,a}$ lifts to a unitary representation of the universal covering of $SL(2,\mathbb{R})$, and can even be extended to a holomorphic semigroup $\Omega_{k,a}$. Our semigroup generalizes the Hermite semigroup studied by R. Howe ($k \equiv 0$, $a = 2$) and the Laguerre semigroup by T. Kobayashi and G. Mano ($k \equiv 0$, $a = 1$). The boundary value of our semigroup $\Omega_{k,a}$ provides us with (k,a)-*generalized Fourier transforms* $\mathcal{F}_{k,a}$, which includes the Dunkl transform $\mathcal{D}_k$ ($a = 2$) and a new unitary operator $\mathcal{H}_k$ ($a = 1$) as a Dunkl-type generalization of the classical Hankel transform. ***To cite this article: S. Ben Saïd et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Transformation de Fourier généralisée $\mathcal{F}_{k,a}$. À l'aide des opérateurs différentiels et aux différences de Dunkl sur $\mathbb{R}^N$, on construit une famille d'actions $\omega_{k,a}$ de l'algèbre de Lie $\mathfrak{sl}(2,\mathbb{R})$ dépendant de deux paramètres k et a. Ici k est une fonction de multiplicité associée aux opérateurs de Dunkl, et $a > 0$ un paramètre d'interpolation entre la représentation de Weil et la représentation minimale du groupe conforme. On montre que $\omega_{k,a}$ s'intègre à une représentation unitaire du revêtement universel du groupe $SL(2,\mathbb{R})$, et se prolonge à un semi-groupe holomorphe $\Omega_{k,a}$. Notre semi-groupe généralise le semi-groupe de Hermite, étudié par R. Howe ($k \equiv 0$, $a = 2$), ainsi que le semi-groupe de Laguerre dû à T. Kobayashi et G. Mano ($k \equiv 0$, $a = 1$). La valeur au bord de notre semi-groupe $\Omega_{k,a}$ donne une transformation de Fourier (k,a)-généralisée $\mathcal{F}_{k,a}$ qui correspond à la transformation de Dunkl pour $a = 2$, et à une nouvelle transformation $\mathcal{H}_k$ pour $a = 1$ qui généralise la transformation de Hankel classique. ***Pour citer cet article : S. Ben Saïd et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

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La transformation de Fourier classique est l'un des objets de base en analyse. On peut la voir comme un élément d'un groupe à un paramètre d'opérateurs unitaires sur $L^2(\mathbb{R})$. Ce groupe se prolonge en un semi-groupe holomorphe (le semi-groupe de Hermite) $I(z)$ engendré par l'opérateur auto-adjoint $\Delta - \|x\|^2$. C'est un semi-groupe holomorphe d'opérateurs bornés, qui dépend d'un nombre complexe z du demi-plan complexe de droite, i.e. $I(z+w) = I(z)I(w)$.

Notre premier objectif est d'analyser le Laplacien de Dunkl Δ_k et de construire une déformation de la situation classique, c'est-à-dire une transformation de Fourier généralisée $\mathcal{F}_{k,a}$ et un semi-groupe holomorphe $\mathcal{I}_{k,a}(z)$ dont le générateur infinitésimal est $\|x\|^{2-a}\Delta_k - \|x\|^a$, agissant sur un espace de Hilbert qui déforme $L^2(\mathbb{R}^N)$. Nous étudions $\mathcal{F}_{k,a}$ et $\mathcal{I}_{k,a}(z)$ dans le cadre des opérateurs intégraux et de la théorie des représentations.

Nos paramètres de déformations sont un paramètre réel $a > 0$ et un paramètre k qui provient de la théorie des opérateurs de Dunkl associés aux groupes de Coxeter finis. Dans notre étude, la dimension N et le nombre complexe z peuvent être considérés comme des paramètres. Nous soulignons que les déformations avec $k \equiv 0$ sont nouvelles. Il est possible qu'elles donnent des généralisations intéressantes du calcul pseudo-différentiel classique. De plus, puisque nous construisons un semi-groupe $\mathcal{I}_{k,a}(z)$ de Hermite–Dunkl et une transformation de Hermite–Dunkl $\mathcal{F}_{k,1}$, notre étude pourrait aussi fournir de nouvelles directions de recherche dans l'analyse des opérateurs de Dunkl, et des processus stochastiques.

Dans le Diagramme 1 on résume quelques propriétés de déformations en indiquant comment des transformations intégrales connues s'intègrent dans notre cadre. En particulier, nous obtenons le semi-groupe de Hermite $I(z)$ [4] ($a = 2$, $k \equiv 0$, et z arbitraire) et le semi-groupe de Laguerre [6] ($a = 1$, $k \equiv 0$, et z arbitraire). La valeur au bord du semi-groupe holomorphe $\mathcal{I}_{k,a}(z)$ de $\operatorname{Re} z > 0$ sur l'axe imaginaire pur donne un sous-groupe à un paramètre d'opérateurs unitaires. La spécialisation $\mathcal{I}_{k,a}(\frac{\pi i}{2})$ est appelée la transformation de Fourier (k,a)-généralisée $\mathcal{F}_{k,a}$ (à un facteur près), qui se réduit à la transformation de Fourier ($a = 2$ et $k \equiv 0$), à la transformation de Dunkl ($a = 2$ et k arbitraire) et à la transformation de Hankel ($a = 1$ et $k \equiv 0$).

Dans cette Note notre ingrédient principal est la construction d'un triplet d'opérateurs différentiels et aux différences qui engendre l'algèbre de Lie de $SL(2,\mathbb{R})$, et le fait que ces représentations infinitésimaux s'intègrent en des représentations unitaires du revêtement universel $\widetilde{SL(2,\mathbb{R})}$. Un autre aspect de notre construction montre un lien entre des représentations minimales unitaires de deux groupes différents.

Les détails seront publiés ultérieurement.

1. Introduction

The classical Fourier transform is one of the most basic objects in analysis; it may be understood as belonging to a one-parameter group of unitary operators on $L^2(\mathbb{R}^N)$, and this group may even be extended holomorphically to a semigroup (the *Hermite semigroup*) $I(z)$ generated by the self-adjoint operator $\Delta - \|x\|^2$. This is a holomorphic semigroup of bounded operators depending on a complex variable z in the complex right half-plane, viz. $I(z+w) = I(z)I(w)$.

The primary aim of our study is to analyze the Dunkl Laplacian Δ_k and to construct a deformation of the classical situation, namely, a generalization $\mathcal{F}_{k,a}$ of the Fourier transform, and the holomorphic semigroup $\mathcal{I}_{k,a}(z)$ with infinitesimal generator $\|x\|^{2-a}\Delta_k - \|x\|^a$, acting on a concrete Hilbert space deforming $L^2(\mathbb{R}^N)$.

Our deformation parameters consist of a real parameter a and a parameter k coming from Dunkl's theory of differential-difference operators; also the dimension N and the complex variable z may be considered as parameters of the theory. We point out that already deformations with $k \equiv 0$ are new.

In Diagram 1 we have summarized some of the deformation properties by indicating the limit behaviour of the holomorphic semigroup $\mathcal{I}_{k,a}(z)$; it is seen how various previous integral transforms fit in our picture. In particular we obtain as special cases the Hermite semigroup $I(z)$ [4] ($a = 2$, $k = 0$ and z arbitrary), and the Laguerre semigroup [6] ($a = 1$, $k \equiv 0$ and z arbitrary). The specialization $\mathcal{I}_{k,a}(\frac{\pi i}{2})$ leads us to a (k,a)-*generalized Fourier transform* $\mathcal{F}_{k,a}$ (up to a phase factor), which reduces to the Fourier transform ($a = 2$ and $k \equiv 0$), the Dunkl transform $\mathcal{D}_k$ ($a = 2$ and k arbitrary), and the Hankel-type transform ($a = 1$ and $k \equiv 0$).

The basic machinery of the present article is a representation theory of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ consisting of differential-difference operators and unitary representations of the universal covering group $G := SL(2,\mathbb{R})^{\sim}$. At the

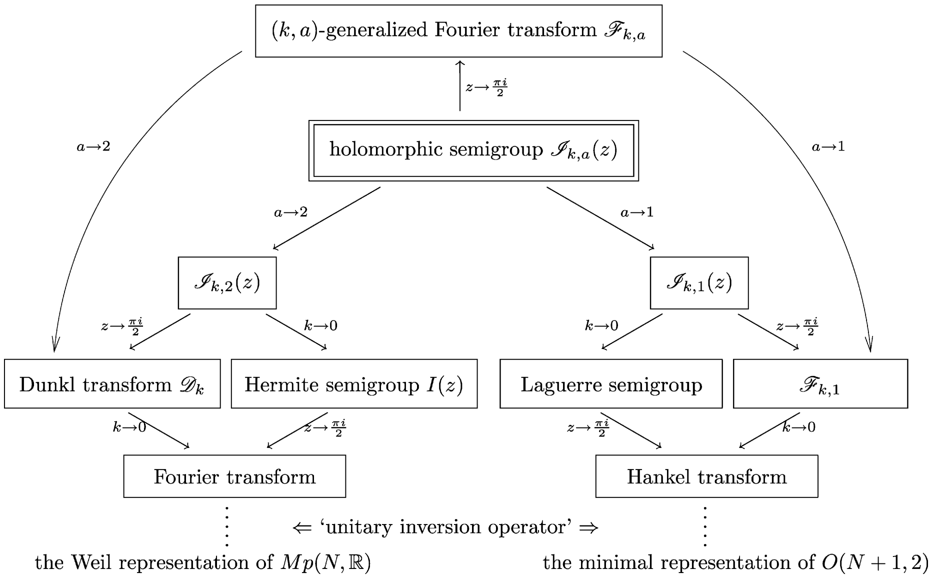

Diagram 1. Special values of holomorphic semigroup $\mathcal{I}_{k,a}(z)$.

special values $a = 1$ and 2, there are further hidden symmetries coming from minimal unitary representations of two reductive groups.

2. Holomorphic semigroup $\mathcal{I}_{k,a}(z)$ with two parameters k and a

Our holomorphic semigroup $\mathcal{I}_{k,a}(z)$ is built on Dunkl operators, see [1] for an excellent exposition. Let $\mathfrak{C}$ be the Coxeter group associated with a reduced root system $\mathcal{R}$ in $\mathbb{R}^N$. For a $\mathfrak{C}$-invariant function $k \equiv (k_\alpha)$ (*multiplicity function*) on $\mathcal{R}$, we set $\langle k\rangle := \frac{1}{2}\sum_{\alpha\in\mathcal{R}} k_\alpha$, and write Δ_k for the Dunkl Laplacian on $\mathbb{R}^N$. This is a differential-difference operator, which reduces to the Euclidean Laplacian when $k \equiv 0$.

We take $a > 0$ to be yet another deformation parameter, and define

$$\Delta_{k,a} := \|x\|^{2-a}\Delta_k - \|x\|^a. \tag{1}$$

We set $\vartheta_{k,a}(x) := \|x\|^{a-2}\prod_{\alpha\in\mathcal{R}} |\langle \alpha, x\rangle|^{k_\alpha}$. In the case $a = 2$ and $k \equiv 0$, $\vartheta_{0,2}(x) \equiv 1$ and $\Delta_{0,2}$ is the Hermite operator on $\mathbb{R}^N$. Here are remarkable properties of our differential-difference operator $\Delta_{k,a}$:

Theorem A. *Suppose $a > 0$ and $a + 2\langle k\rangle + N - 2 > 0$.*

1) *$\Delta_{k,a}$ extends to a self-adjoint operator on the Hilbert space $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)\,\mathrm{d}x)$.*
2) *There is no continuous spectrum of $\Delta_{k,a}$. Furthermore, all the discrete spectra of $\Delta_{k,a}$ are negative.*

The (k, a)-*generalized Laguerre semigroup* $\mathcal{I}_{k,a}(z)$ is defined as

$$\mathcal{I}_{k,a}(z) := \exp\left(\frac{z}{a}\Delta_{k,a}\right), \quad \text{for } \operatorname{Re} z \geqslant 0. \tag{2}$$

We note that $\mathcal{I}_{0,2}(z)$ is the Hermite semigroup $I(z)$ given by the Mehler kernel [4], and $\mathcal{I}_{0,1}(z)$ is the Laguerre semigroup whose kernel is given in terms of the Bessel function [6].

Theorem B. *Suppose $a > 0$ and $a + 2\langle k\rangle + N - 2 > 0$.*

1) *$\mathcal{I}_{k,a}(z)$ for $\operatorname{Re} z > 0$ is a holomorphic semigroup consisting of Hilbert–Schmidt operators.*
2) *$\mathcal{I}_{k,a}(z)$ is a one-parameter group of unitary operators on the imaginary axis $\operatorname{Re} z = 0$.*

3. (k,a)-generalized Fourier transforms $\mathcal{F}_{k,a}$

Theorem B 2) says that the 'boundary value' of the holomorphic semigroup $\mathcal{I}_{k,a}(z)$ gives a one-parameter family of unitary operators. The underlying idea may be interpreted as a descendant of Sato's hyperfunction theory [9] and also that of the Gelfand–Gindikin program (see [3] and references therein) for unitary representations of real reductive groups.

The case $z=0$ gives the identity operator, namely, $\mathcal{I}_{k,a}(0)=\mathrm{id}$. The particularly interesting case is when $z=\frac{\pi i}{2}$. We set $c := \exp(i\pi\frac{N+2\langle k\rangle+a-2}{2a})$ (phase factor), and define

$$\mathcal{F}_{k,a} := c\,\mathcal{I}_{k,a}\left(\frac{\pi i}{2}\right) = c\exp\left(\frac{\pi i}{2a}\left(\|x\|^{2-a}\Delta_k - \|x\|^a\right)\right) \quad \big((k,a)\text{-generalized Fourier transform}\big).$$

Then, this operator $\mathcal{F}_{k,a}$ for general a and k satisfies the following significant properties:

Theorem C. *Suppose $a>0$ and $a+2\langle k\rangle+N-2>0$.*

1) *$\mathcal{F}_{k,a}$ is a unitary operator on $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)\,\mathrm{d}x)$.*
2) *$\mathcal{F}_{k,a}\circ E = -(E+N+2\langle k\rangle+a-2)\circ\mathcal{F}_{k,a}$. Here, $E=\sum_{j=1}^N x_j\partial_j$ is the Euler operator.*
3) *$\mathcal{F}_{k,a}\circ\|x\|^a = -\|x\|^{2-a}\Delta_k\circ\mathcal{F}_{k,a}$, $\mathcal{F}_{k,a}\circ(\|x\|^{2-a}\Delta_k) = -\|x\|^a\circ\mathcal{F}_{k,a}$.*
4) *$\mathcal{F}_{k,a}$ is of finite order if and only if $a\in\mathbb{Q}$. Its order is $2p$ if $\frac{p}{a}$ is an integer prime to p.*

As indicated in Diagram 1, $\mathcal{F}_{k,a}$ reduces to the Euclidean Fourier transform $\mathcal{F}$ on $\mathbb{R}^N$ if $k\equiv 0$ and $a=2$; to the Dunkl transform $\mathcal{D}_k$ introduced by C. Dunkl himself if $k>0$ and $a=2$. The unitary operator $\mathcal{F}_{0,1}$ arises as the *unitary inversion operator* of the Schrödinger model of the minimal representation of the conformal group $O(N+1,2)$ (see [6]). Its Dunkl analogue $\mathcal{H}_k := \mathcal{F}_{k,1}$ is an involutive unitary operator on $L^2(\mathbb{R}^N, \vartheta_{k,1}(x)\,\mathrm{d}x)$ whose kernel is explicitly given by using the formula (4) below.

Our study also contributes to the theory of special functions, in particular orthogonal polynomials; indeed we derive several new identities, for example, the (k,a)-deformation of the classical Hecke identity where the Gaussian function and harmonic polynomials in the classical setting are replaced respectively with $\exp(-\frac{1}{a}\|x\|^a)$ and polynomials annihilated by the Dunkl Laplacian. We also have:

Theorem D *(Heisenberg type inequality). Let $\|\ \|_k$ denote the norm on $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)\,\mathrm{d}x)$. Then,*

$$\big\|\|x\|^{\frac{a}{2}}f(x)\big\|_k\big\|\|\xi\|^{\frac{a}{2}}\mathcal{F}_{k,a}f(\xi)\big\|_k \geqslant \frac{2\langle k\rangle+N+a-2}{2}\|f(x)\|_k^2, \quad \textit{for any } f\in L^2\big(\mathbb{R}^N, \vartheta_{k,a}(x)\,\mathrm{d}x\big).$$

The equality holds if and only if f is a scalar multiple of $\exp(-c\|x\|^a)$ for some $c>0$.

This inequality was previously known by Rösler [8] for the $a=2$ case (i.e. the Dunkl transform $\mathcal{D}_k$). For $a=1$, we may think of the function $\exp(-c\|x\|^a)$ as a ground state in physics terms; indeed when $a=c=1$ it is exactly the wave function for the Hydrogen atom with the lowest energy.

4. Integral representation of $\mathcal{I}_{k,a}(z)$ and $\mathcal{F}_{k,a}$

The Euclidean Fourier transform is given by the integral against the kernel $(2\pi)^{-\frac{N}{2}}e^{-i\langle x,\xi\rangle}$, and the Hermite semigroup is given by the Mehler kernel. Generalizing this, we find the integral expression of the (k,a)-generalized Fourier transform $\mathcal{F}_{k,a}$ and the holomorphic semigroup $\mathcal{I}_{k,a}(z)$.

In this section, we assume $k\geqslant 0$. *Dunkl's intertwining operator* V_k is a topological linear isomorphism of the space of continuous functions on $\mathbb{R}^N$, which intertwines the Dunkl operators and the directional derivatives [1]. For a continuous function $h(t)$ of one variable, we set $h_y(\cdot) := h(\langle\cdot,y\rangle)$ $(y\in\mathbb{R}^N)$, and define $(\widetilde{V}_k h)(x,y) := (V_k h_y)(x)$. Then $(\widetilde{V}_k h)(x,y)$ is a continuous function of (x,y). Further, $(\widetilde{V}_k h)(x,y) = (\widetilde{V}_k h)(y,x)$. We note that $(\widetilde{V}_k h)(x,y) = h(\langle x,y\rangle)$ if $k\equiv 0$.

Let $\widetilde{I}_\lambda(w) = (\frac{w}{2})^{-\lambda} I_\lambda(w)$ be the normalized I-Bessel function, and $C_m^\nu(t)$ the Gegenbauer polynomial. Then the following series converges absolutely on $\{(b, \nu, w, t) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{C} \times [-1, 1]\colon\ 1 + b\nu > 0\}$:

$$\mathcal{J}(b, \nu; w; t) = \frac{\Gamma(b\nu + 1)}{\nu} \sum_{m=0}^{\infty} (m + \nu) \left(\frac{w}{2}\right)^{bm} \widetilde{I}_{b(m+\nu)}(w) C_m^\nu(t). \tag{3}$$

The special values at $b = 1, 2$ are given by

$$\mathcal{J}(1, \nu; w; t) = e^{wt}, \qquad \mathcal{J}(2, \nu; w; t) = \Gamma\left(\nu + \frac{1}{2}\right) \widetilde{I}_{\nu-\frac{1}{2}}\left(\frac{w(1+t)^{1/2}}{\sqrt{2}}\right). \tag{4}$$

We introduce the following continuous function on $[-1, 1]$ with parameters $r, s > 0$ and $z \notin i\pi\mathbb{Z}$:

$$h_{k,a}(r, s; z; t) = \frac{\exp(-\frac{1}{a}(r^a + s^a)\coth(z))}{\sinh(z)^{\frac{2\langle k\rangle + N + a - 2}{a}}} \mathcal{J}\left(\frac{2}{a}, \frac{2\langle k\rangle + N - 2}{2}; \frac{2(rs)^{\frac{a}{2}}}{a \sinh(z)}; t\right),$$

and define $\Lambda_{k,a}(x, y; z) = \widetilde{V}_k(h_{k,a}(r, s; z; \cdot))(\omega, \eta)$ in the polar coordinates $x = r\omega$, $y = s\eta$. Then, we have:

Theorem E. *Suppose $a > 0$ and k is a non-negative multiplicity function. Suppose $\operatorname{Re} z \geqslant 0$ and $z \notin i\pi\mathbb{Z}$. Then, the holomorphic semigroup $\mathcal{I}_{k,a}(z)$ (see (2)) is given by*

$$\mathcal{I}_{k,a}(z) f(x) = c_{k,a} \int_{\mathbb{R}^N} f(y) \Lambda_{k,a}(x, y; z) \vartheta_{k,a}(y)\, \mathrm{d}y. \tag{5}$$

Here, the normalizing constant $c_{k,q} := (\int_{\mathbb{R}^N} \exp(-\frac{1}{a}\|x\|^a)\vartheta_{k,a}(x)\, \mathrm{d}x)^{-1}$ is found explicitly by Selberg, Macdonald, Heckman, Opdam, and Etingof (see e.g. [1]).

5. Hidden symmetries in the Hilbert space $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)\, \mathrm{d}x)$

We introduce the following differential-difference operators on $\mathbb{R}^N \setminus \{0\}$ by

$$\mathbb{E}_{k,a}^+ := \frac{i}{a}\|x\|^a, \qquad \mathbb{E}_{k,a}^- := \frac{i}{a}\|x\|^{2-a}\Delta_k, \qquad \mathbb{H}_{k,a} := \frac{2}{a}\sum_{i=1}^N x_i \partial_i + \frac{N + 2\langle k\rangle + a - 2}{a}.$$

Lemma F. *The differential-difference operators $\{\mathbb{H}_{k,a}, \mathbb{E}_{k,a}^+, \mathbb{E}_{k,a}^-\}$ form an $\mathfrak{sl}_2$-triple for any multiplicity-function k and any non-zero complex number a.*

Special cases of Lemma F was previously known: the case $k \equiv 0$ and $a = 2$ is the classical harmonic $\mathfrak{sl}_2$-triple (e.g. Howe [4]), the case $k > 0$ and $a = 2$ by Heckman [2], and $k \equiv 0$ and $a = 1$ by Kobayashi and Mano [6]. Our operator $\Delta_{k,a} = \frac{i}{a}(\mathbb{E}_{k,a}^+ - \mathbb{E}_{k,a}^-)$ can be interpreted as an element of $\mathfrak{sl}(2, \mathbb{R})$.

Lemma F fits nicely into the theory of discretely decomposable representations of reductive groups [5]:

Theorem G. *If $a > 0$ and $a + 2\langle k\rangle + N - 2 > 0$, then the representation of $\mathfrak{sl}(2, \mathbb{R})$ lifts to a unitary representation of the simply-connected group G on $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)\, \mathrm{d}x)$. It is discretely decomposable, and commutes the obvious action of the Coxeter group $\mathfrak{C}$.*

This unitary representation is the key to the proof of Theorems A to E. The special case $N = 1$ and $k \equiv 0$ recovers Kostant's realization [7] of highest weight representations of G. For $N \geqslant 2$, it contains countably many irreducible representations of G, which we can find explicitly for general k and a.

Theorem G asserts that the Hilbert space $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)\, \mathrm{d}x)$ has a symmetry of the direct product group $\mathfrak{C} \times G$ for all k and a. This symmetry becomes larger for special values of k and a as shown in Diagram 2.

For $a = 2$, this unitary representation of $Mp(N, \mathbb{R})$ is nothing but the Weil representation. For $a = 1$, the unitary representation of $O(N+1, 2)^{\sim}$ (a double covering of the conformal group) on $L^2(\mathbb{R}^N, \|x\|^{-1}\, \mathrm{d}x)$ is irreducible. Both

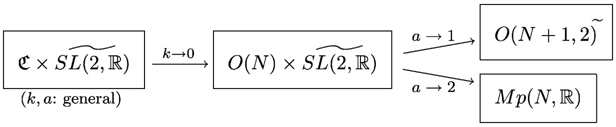

Diagram 2. Hidden symmetries in $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)\,\mathrm{d}x)$.

of them are so-called the minimal representations and, in particular, they attain the minimum of their Gelfand–Kirillov dimensions among the unitary dual. In this sense, our continuous parameter $a > 0$ interpolates the L^2-models of two minimal representations of different reductive groups by keeping smaller symmetries. The (k, a)-generalized Fourier transform $\mathcal{F}_{k,a}$ ($k \equiv 0$, $a = 1, 2$) arise as the unitary operators (up to phase factors) corresponding to the longest Weyl group elements.

Detailed proofs will appear elsewhere.

References

[1] C.F. Dunkl, Reflection groups in analysis and applications, Japan J. Math. 3 (2008) 215–246.
[2] G.J. Heckman, A remark on the Dunkl differential-difference operators, in: W. Barker, P. Sally (Eds.), Harmonic Analysis on Reductive Groups, in: Progr. Math., vol. 101, Birkhäuser, 1991, pp. 181–191.
[3] J. Hilgert, K.-H. Neeb, Positive definite spherical functions on Olshanski domains, Trans. Amer. Math. Soc. 352 (2000) 1345–1380.
[4] R. Howe, The oscillator semigroup, in: Proc. Sympos. Pure Math., vol. 48, Amer. Math. Soc., 1988, pp. 61–132.
[5] T. Kobayashi, Discrete decomposability of the restriction of $A_{\mathfrak{q}}(\lambda)$ with respect to reductive subgroups, Invent. Math. 117 (1994) 181–205; Part II, Ann. Math. 147 (1998) 709–729; Part III, Invent. Math. 131 (1998) 229–256.
[6] T. Kobayashi, G. Mano, The inversion formula and holomorphic extension of the minimal representation of the conformal group, in: Harmonic Analysis, Group Representations, Automorphic Forms and Invariant Theory: In Honor of Roger Howe, World Scientific, 2007, pp. 159–223, cf. arXiv:math.RT/0607007.
[7] B. Kostant, On Laguerre polynomials, Bessel functions, and Hankel transform and a series in the unitary dual of the simply-connected covering group of $SL(2,\mathbb{R})$, Represent. Theory 4 (2000) 181–224.
[8] M. Rösler, An uncertainty principle for the Dunkl transform, Bull. Austral. Math. Soc. 59 (1999) 353–360.
[9] M. Sato, Theory of hyperfunctions I, J. Fac. Sci. Univ. Tokyo 8 (1959) 139–193.