Harmonic Analysis

Boundedness of the square function and rectifiability ☆

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Abstract

Following a recent paper [X. Tolsa, Principal values for Riesz transforms and rectifiability, J. Funct. Anal. 254 (7) (2008) 1811–1863] we show that the finiteness of the square function associated with the Riesz transforms with respect to Hausdorff measure H^n (n is an integer) on a set E, implies that E is rectifiable. *To cite this article: S. Mayboroda, A. Volberg, C. R. Acad. Sci. Paris, Ser. I 347 (2009).*
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Résumé

Une fonction carrée de la transformation de Riesz et rectifiabilité. On peut modifier l'article récent [X. Tolsa, Principal values for Riesz transforms and rectifiability, J. Funct. Anal. 254 (7) (2008) 1811–1863] pour démontrer que la convergence de la fonction carrée associée aux transformations de Riesz de mesure de Hausdorff H^n (n est un nombre entier) sur un compact E implique que E est rectifiable. *Pour citer cet article : S. Mayboroda, A. Volberg, C. R. Acad. Sci. Paris, Ser. I 347 (2009).*
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Soit E un compact de $\mathbb{R}^d$ tel que $0 < H^n(E) < \infty$, $0 < n \leqslant d$, $n \in \mathbb{Z}$.

Les n-densités supérieure et inférieure d'une mesure de Borel μ en un point x sont respectivement données par

$$\theta_\mu^{n,*}(x) := \limsup_{r \to 0} \frac{\mu(B(x,r))}{r^n} \quad \text{et} \quad \theta_{\mu,*}^n(x) := \liminf_{r \to 0} \frac{\mu(B(x,r))}{r^n}, \tag{1}$$

supposons pour le moment que $\mu = H^n | E$, $0 < H^n(E) < \infty$. Pour $d = 2, n = 1$ on sait que

$$\sup_{\varepsilon > 0} \left| R_\varepsilon^1 \mu(x) \right| < \infty \quad \mu\text{-p.p. } x \in E, \tag{2}$$

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implique la rectifiabilité de E. C'est un résultat combiné des travaux de Tolsa et Nazarov–Treil–Volberg. Un problème bien connu et non encore réglé d'analyse non-homogène dans $\mathbb{R}^d$ est d'aboutir à la même conclusion à partir de l'hypothèse que ($\mu := H^n|E$, $0 < d < n$, $d > 2$)

$$\sup_{\varepsilon > 0} \left| R_\varepsilon^n \mu(x) \right| < \infty \quad \mu\text{-p.p. } x \in E. \tag{3}$$

Même le cas de $n = 1$, $d > 2$ est ouvert. Ici, pour une mesure de Borel μ dans $\mathbb{R}^d$ et un nombre entier $n \in (0, d]$, la transformation de Riesz de μ est définie par

$$R^n \mu(x) := \int \frac{x - y}{|x - y|^{n+1}} \, d\mu(y), \quad x \notin \operatorname{supp} \mu, \tag{4}$$

et la transformation de Riesz tronquée est

$$R_\varepsilon^n \mu(x) := \int_{|x-y|>\varepsilon} \frac{x - y}{|x - y|^{n+1}} \, d\mu(y), \quad x \in \mathbb{R}^d, \quad \varepsilon > 0. \tag{5}$$

Si on remplace l'hypothèse (3) par le fait que l'opérateur de Riesz est borné dans $L^2(\mu)$:

$$\sup_{\varepsilon > 0} \left\| R_\varepsilon^n : L^2(\mu) \to L^2(\mu) \right\| < \infty \tag{6}$$

on arrive à un problème pratiquement équivalent. Cette question est connue comme un problème de Guy David, problème largement ouvert même pour le cas « homogène » quand la densité satisfait $0 < c < \theta_*^n(\mu)(x) \leqslant \theta^{n,*}(\mu)(x) < C < \infty$ uniformément sur E.

Selon la théorie de David et Semmes, si on remplace un opérateur de Riesz en (6) par tous les opérateurs de Calderón–Zygmund d'une certaine classe, on obtient la rectifiabilité de E (E contient « les gros morceaux de graphes lipschitziens »).

Supposons que seuls les opérateurs de Calderón–Zygmund obtenus par la procédure de Littlewood–Paley soient bornés dans $L^2(\mu)$. Il s'agit d'une classe beaucoup plus petite que la classe de David–Semmes. On démontre ici que cette classe effectivement est suffisante pour conclure que E contient « les gros morceaux de graphes lipschitziens ». Autrement dit, nous prouvons qu'il est suffisant d'avoir une seul opérateur borné dans $L^2(\mu)$, et que cet opérateur est la fonction carrée de type de Lusin, qui, entre parenthèses, est la moyenne des opérateurs de Littlewood–Paley.

Malheureusement, on ne sait pas comment passer de (6) au fait que la fonction carrée soit bornée, ou aux faits que les opérateurs de Littlewood–Paley soient bornés.

Le principal résultat de cette communication est le suivant :

Théorème 0.1. *Soit μ une mesure de Radon finie sur $\mathbb{R}^d$ avec*

$$0 < \theta_\mu^{n,*}(x) < \infty, \quad \mu\text{-p.p. } x \in \mathbb{R}^d. \tag{7}$$

Soit $n \in (0, d]$ un nombre entier, et supposons que

$$S\mu(x) := \left(\int_0^1 \left| R_{u,2u}^n \mu(x) \right|^2 \frac{du}{u} \right)^{1/2} < \infty, \quad \mu\text{-p.p. } x \in \mathbb{R}^d. \tag{8}$$

Alors, il existe un nombre dénombrable de surfaces rectifiables Γ_n de dimension n tel que $\mu(\mathbb{R}^d \setminus \bigcup_n \Gamma_n) = 0$.

Un élément essentiel de notre preuve est de savoir extraire la proposition suivante à partir de (8)

$$\exists \, F \subset \operatorname{supp}(\mu), \, \mu(F) > 0 : \|S\|_{L^2(\mu|_F) \to L^2(\mu|_F)} \leqslant M_2. \tag{9}$$

Proposition 0.1. *Soit μ une mesure positive Radon sur $\mathbb{R}^d$, soit $0 < \theta_\mu^{n,*}(x) < \infty$, μ-p.p., $n \in \mathbb{N}$, $n \leqslant d$, et soit $S\mu(x) < \infty$, μ-p.p. Alors, pour chaque $\delta > 0$ il existe un compact $E \subset \mathbb{R}^d$ tel que S est un opérateur borné dans $L^2(\mu|_E)$, et $\mu(\mathbb{R}^d \setminus E) \leqslant \delta$.*

Pour conclure que E est rectifiable à partir de l'hypothèse (3) ou à partir de l'hypothèse (6) suffise de démontrer

La conjecture. Il existe une partie $E' \subset E = \operatorname{supp} \mu$, $\mu(E') > 0$ telle que

$$\lim_{\varepsilon \to 0} R^n_{\varepsilon, 2\varepsilon} \mu(x) = 0 \quad \mu\text{-p.p. } x \in E'. \tag{10}$$

1. Introduction

For a Borel measure μ in $\mathbb{R}^d$ and $s \in (0, d]$ the s-Riesz transform of μ is defined as

$$R^s \mu(x) := \int \frac{x - y}{|x - y|^{s+1}} \, d\mu(y), \quad x \notin \operatorname{supp} \mu, \tag{11}$$

and the truncated Riesz transform is given by

$$R^s_\varepsilon \mu(x) := \int\limits_{|x-y|>\varepsilon} \frac{x - y}{|x - y|^{s+1}} \, d\mu(y), \quad R^s_{\varepsilon, \eta} \mu(x) := R^s_\eta \mu(x) - R^s_\varepsilon \mu(x), \tag{12}$$

where $x \in \mathbb{R}^d$, $\eta > \varepsilon > 0$.

One says that the set $E \subset \mathbb{R}^d$ is n-rectifiable, $n \in \mathbb{N}$, if there is a countable family of n-dimensional C^1 submanifolds $\{M_i\}_{i \geqslant 1}$ such that $H^n(E \setminus \bigcup_i M_i) = 0$, where H^n stands for the n-dimensional Hausdorff measure.

The main result of this Note is the following:

Theorem 1.1. *Let μ be a finite Radon measure in $\mathbb{R}^d$. Assume that for some $s \in (0, d]$ there is a set $E \subset \mathbb{R}^d$ with the property that*

$$0 < \theta^{s,*}_\mu(x) < \infty \quad \text{for all } x \in E, \tag{13}$$

and the square function

$$S\mu(x) := \left(\int\limits_0^\infty \left| R^s_{t, 2t} \mu(x) \right|^2 \frac{dt}{t} \right)^{1/2} < \infty \quad \text{for all } x \in E. \tag{14}$$

Then s is an integer and E is s-rectifiable.

In particular, if E is a compact set in $\mathbb{R}^d$ with $0 < H^s(E) < \infty$ and (14) is satisfied, then s is an integer and E is s-rectifiable.

Here and below, the upper s-dimensional density of μ at x is given by $\theta^{s,*}_\mu(x) := \limsup_{r \to 0} \frac{\mu(B(x,r))}{r^s}$, where $B(x, r)$ is the ball of radius $r > 0$ centered at $x \in \mathbb{R}^d$.

In [1,2] David and Semmes showed, under certain assumptions on the measure μ, that the L^2 boundedness of *all* Calderón–Zygmund singular integral operators implies that s is an integer and μ is s-rectifiable. Our main result assures that it is, in fact, sufficient to assume pointwise boundedness of a single operator, namely, the square function, in order to arrive to the same conclusion. Alternatively, one could view (14) *almost* as a condition

$$\mathbb{E} \left| \sum_{k \in \mathbb{Z}} \varepsilon_k R^s_{2^{-k}, 2^{-k+1}} \mu(x) \right| < \infty \quad \mu\text{-a.e. } x \in \mathbb{R}^m, \tag{15}$$

where ε_k are independent random variables taking the values -1 and 1 with probability $1/2$ each. Therefore, roughly speaking, the boundedness of the singular integrals of the type $\sum_{k=0}^\infty \varepsilon_k R^s_{2^{-k}, 2^{-k+1}} \mu(x)$ already guarantees that μ is s-rectifiable, and the corresponding class of operators is much smaller than that of David and Semmes.

Let us point out that we have already proved the fact that (13)–(14) imply that s is an integer in [5]. The present Note, concentrating on the issue of rectifiability, is a sequel to the aforementioned work.

Ultimately, one would like to show that the conclusion of Theorem 1.1 holds purely under the assumption that the Riesz transforms $R^s_\varepsilon \mu$ are bounded in L^2 uniformly in $\varepsilon > 0$, or under the condition

$$\sup_{\varepsilon > 0} \left| R^s_\varepsilon \mu(x) \right| < \infty \quad \mu\text{-a.e. } x \in \mathbb{R}^d. \tag{16}$$

This is a long-standing open problem, known as a conjecture of Guy David. At the moment, it has been resolved only for $0 < s \leqslant 1$ [9,8]. The proof heavily relies on the curvature of measure estimates from [6], which are decisively restricted to lower dimensions. However, in the presence of Theorem 1.1, the problem of Guy David in all dimensions is essentially reduced to the passage from (16) to (14).

Our methodology builds on recent advances in [10], where the author proved that the existence of $p.v.R^n \mu(x) := \lim_{\varepsilon \to 0} R_\varepsilon^n \mu(x)$ for μ-a.e. $x \in \mathbb{R}^d$ entails that μ is n-rectifiable. Analogously to the argument in [10], one of the main ingredients in our proof is an estimate for the L^2 norm of the square function, on the graph of a Lipschitz function A, in terms of $\|\nabla A\|_2$. It compensates for the lack of the curvature of measure techniques. Having this at hand, the matters are further reduced to Léger's construction of Lipschitz graphs [4]. In the present Note we outline the main stages of the proof, and the details will appear elsewhere.

2. L^2 estimates on a Lipschitz graph

Let us start by introducing some extra notation. As we already mentioned, the case of the non-integer dimension was treated in [5], and here we focus on $s = n$, $n \in \mathbb{N}$, $n \leqslant d$. Let Π stand for the projection $(x_1, \ldots, x_n, \ldots, x_d) \to (x_1, \ldots, x_n, 0, \ldots, 0)$, $\Pi^\perp := I - \Pi$.

Furthermore, let $R^j \mu(x)$, $k \in \mathbb{Z}$, $x \in \mathbb{R}^d$, be a version of doubly truncated Riesz transform, defined, for example, as follows. If ψ^0 is a non-increasing radial C^∞ function with $\chi(B(0, 1/2)) \leqslant \psi^0 \leqslant \chi(B(0, 2))$ we set $\psi_j(z) := \psi^0(2^j z) - \psi^0(2^{j+1} z)$, $j \in \mathbb{Z}$, $z \in \mathbb{R}^d$, so that each ψ_j is non-negative, $\operatorname{supp} \psi_j \subset B(0, 2^{-j+1}) \setminus B(0, 2^{-j-2})$ and $\sum_{j \in \mathbb{Z}} \psi_j = 1$ in $\mathbb{R}^d \setminus \{0\}$. Then $R^j \mu$ is an operator defined analogously to (11) with the kernel given by $\psi_j(x - y)\frac{x-y}{|x-y|^{s+1}}$.

It is not hard to show that the condition (14) implies that the discrete version of the square function

$$\widetilde{S}\mu(x) := \left(\sum_{j \in \mathbb{Z}} \left| R^j \mu(x) \right|^2 \right)^{1/2} < \infty \quad \text{for all } x \in E. \tag{17}$$

In the course of the proof we will also employ the notation $\widetilde{S}^\perp \mu := (\sum_{j \in \mathbb{Z}} |\Pi^\perp (R^j \mu)|^2)^{1/2}$ and $S^\perp \mu = (\int_0^\infty |\Pi^\perp (R_{t,2t}^n \mu)|^2 \frac{dt}{t})^{1/2}$.

Our reasoning follows closely that in [10], replacing R's by S's, and one of the cornerstones of our argument is the following L^2 bound for the square function on a Lipschitz graph:

Theorem 2.1. *Let Γ be an n-dimensional Lipschitz graph $\Gamma = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^{d-n} : y = A(x)\}$ and let $d\mu = g(z) dH_{|\Gamma}^n(z)$ with $C_1^{-1} \leqslant g(z) \leqslant C_1$ for some $C_1 > 0$ and all $z \in \Gamma$. Assume further that A is compactly supported, $\|g - 1\|_2 \leqslant C_2 \|\nabla A\|_2$, and $\|\nabla A\|_\infty \leqslant \varepsilon_0$. Then for $\varepsilon_0 = \varepsilon_0(C_2)$ sufficiently small*

$$\left\| S^\perp \mu \right\|_{L^2(\mu)} \approx \|S\mu\|_{L^2(\mu)} \approx \|\nabla A\|_2. \tag{18}$$

Sketch of proof. The upper estimate $\|S\mu\|_{L^2(\mu)} \leqslant C\|\nabla A\|_2$ does not require the smallness of ε_0, and follows directly by combining (5.10) in [11] with the argument of Lemma 3.1 in [10]. The lower bound, $\|S^\perp \mu\|_{L^2(\mu)} \geqslant C\|\nabla A\|_2$, is more involved.

Observe that $\Pi_\sharp \mu = \rho(x) dx$ with $\rho(x) = g(x) J\widetilde{A}(x)$, where $\widetilde{A}(x) = (x, A(x))$, $x \in \mathbb{R}^n$, and $J\widetilde{A}$ stands for the n-dimensional Jacobian of $\widetilde{A}$. Take now μ_0 with $\operatorname{supp} \mu_0 \subset \Gamma$ such that $\Pi_\sharp \mu_0 = dx$. Then $d\mu(x) - d\mu_0(x) = (h(x) - 1) d\mu_0(x)$, $h = \rho(\Pi(x))$, and one can show that $\|\rho - 1\|_2 \leqslant C\|\nabla A\|_2$, so that $\|h - 1\|_2 \leqslant C\|\nabla A\|_2$.

Carefully tracking the argument in Chapter 5 of [10] we deduce that for such a measure μ_0

$$\left\| S^\perp \mu_0 \right\|_{L^2(\mu_0)}^2 \approx \sum_{j \in \mathbb{Z}} \left\| \Pi^\perp (R^j \mu_0) \right\|_{L^2(\mu_0)}^2 \approx \|\nabla A\|_2^2, \tag{19}$$

hence, it remains to estimate the difference between $\|S^\perp \mu_0\|_{L^2(\mu_0)}$ and $\|S^\perp \mu\|_{L^2(\mu)}$. However,

$$\left| \left\| S^\perp \mu_0 \right\|_{L^2(\mu_0)} - \left\| S^\perp \mu \right\|_{L^2(\mu_0)} \right| \leqslant \left\| S^\perp (\mu - \mu_0) \right\|_{L^2(\mu_0)} = \left\| S^\perp \big((h - 1) d\mu_0 \big) \right\|_{L^2(\mu_0)}. \tag{20}$$

Furthermore, when $\Pi_\sharp \mu_0 = dx$, one can directly show that the operator $S^\perp$ is bounded in $L^2(\mu_0)$, with the norm controlled by $C\|\nabla A\|_\infty$. Hence, the right-hand side of (20) is bounded from above by $C\|\nabla A\|_\infty \|h -$

$1\|_{L^2(\mu_0)} \leqslant \varepsilon_0\|\nabla A\|_2$. Then, if ε_0 is sufficiently small, (19)–(20) lead to $\|S^\perp\mu\|_{L^2(\mu_0)} \approx \|\nabla A\|_2$, which implies that $\|S^\perp\mu\|_{L^2(\mu)} \approx \|\nabla A\|_2$ since $g(x) \approx h(x) \approx 1$. □

3. From E to the construction of Lipschitz graphs

Theorem 1.1 can be reduced to the following proposition mimicking the main lemma of [10]:

Proposition 3.1. *Let μ be a finite Borel measure in $\mathbb{R}^d$. Assume that there is a closed ball $B_0 = \overline{B}(x_0, r_0)$ and a compact set $F \subset 10B_0$ with $x_0 \in F$ such that for some positive constants M_1, M_2, δ_1, δ_2:*

(i) *$\mu(8B_0) = c_n 8^n r_0^n$ and $\mu(10B_0 \setminus F) \leqslant \delta_1\mu(B_0)$,*
(ii) *$\mu(B(x,r)) \leqslant M_1 r^n$ for all $x \in F$, $r > 0$, and $\mu(B(x,r)) \leqslant c_n(1 + \delta_1)r^n$ for all $x \in F$ and $0 < r \leqslant 100r_0$,*
(iii) *$\|S\|_{L^2(\mu|_F), L^2(\mu|_F)} \leqslant M_2$,*
(iv) *$|\widetilde{R}_{\varepsilon,2\varepsilon}\mu(x)| + |\widehat{R}_{\varepsilon,2\varepsilon}\mu(x)| \leqslant \delta_2$ for all $x \in F$ and $0 < \varepsilon < \delta_2^{-2}r_0$.*

If $\delta_1 = \delta_1(M_2)$ and $\delta_2 = \delta_2(M_1, M_2)$ are small enough, then there exists an n-dimensional Lipschitz graph Γ with the property that

$$\mu(\Gamma \cap F \cap B_0) \geqslant \frac{9}{10}c_n r_0^n. \tag{21}$$

Here,

$$\widetilde{R}_\varepsilon\mu(x) := \int \frac{x-y}{(\varepsilon^2 + |x-y|)^{\frac{n+1}{2}}} \, d\mu(y), \quad \widehat{R}_\varepsilon\mu(x) = \int \psi\left(\frac{x-y}{\varepsilon}\right)\frac{x-y}{|x-y|^{n+1}} \, d\mu(y), \tag{22}$$

for some C^∞ function ψ with $\chi_{\mathbb{R}^d \setminus B(0,1)} \leqslant \psi \leqslant \chi_{\mathbb{R}^d \setminus B(0,1/2)}$, and $\widetilde{R}_{\varepsilon,2\varepsilon}\mu = \widetilde{R}_{2\varepsilon}\mu - \widetilde{R}_\varepsilon\mu$, $\widehat{R}_{\varepsilon,2\varepsilon}\mu = \widehat{R}_{2\varepsilon}\mu - \widehat{R}_\varepsilon\mu$, $\varepsilon > 0$.

The reduction from Theorem 1.1 to Proposition 3.1 follows the general lines of the argument in Chapter 7 of [10]. An important new element is the need to extract the sets satisfying the condition (iii) above. This is also interesting on its own right, and is stated in the following proposition.

Proposition 3.2. *Let μ be a positive finite Radon measure on $\mathbb{R}^d$, satisfying $0 < \theta_\mu^{n,*}(x) < \infty$ for μ-a.e. $x \in \mathbb{R}^d$, $n \in \mathbb{N}$, $n \leqslant d$, and such that $S\mu(x) < \infty$ for μ-a.e. $x \in \mathbb{R}^d$. Then for every $\delta > 0$ there is a compact set $E \subset \mathbb{R}^d$ such that S is bounded in $L^2(\mu|_E)$ and $\mu(\mathbb{R}^d \setminus E) \leqslant \delta$.*

Sketch of proof. Following the techniques in [12] and [9], we prove Proposition 3.2 via reduction to the operators with suppressed kernels. To this end, let us denote by S_ε the square function defined analogously to (14) with the integration over $t \in (\varepsilon, \infty)$. Also, let S^θ be an analogue of (14) with the integrand given by $R_{t,2t}^\theta$, where θ is a non-negative Lipschitz function with Lipschitz constant less than or equal to 1, and

$$R_{t,2t}^\theta\mu(x) = \int\limits_{t \leqslant |x-y| \leqslant 2t} k_\theta(x,y) \, d\mu(y), \quad k_\theta(x,y) = k(x,y)\frac{1}{1 + k^2(x,y)\theta(x)\theta(y)}, \tag{23}$$

with $k(x,y) = \frac{x-y}{|x-y|^{n+1}}$ being the kernel of the Riesz transform.

First, we show that the square function with the suppressed kernel $S^\theta\mu$ is controlled by $S_\varepsilon\mu$, in the following sense. If for some $x \in \mathbb{R}^d$ and $r_0 \geqslant 0$

$$\mu\big(B(x,r)\big) \leqslant C_0 r^n \quad \forall r \geqslant r_0, \quad \text{and} \quad \big|S_\varepsilon\mu(x)\big| \leqslant C_1 \quad \forall \varepsilon \geqslant r_0, \tag{24}$$

then $|S_\theta\mu(x)| \leqslant C_2(C_0, C_1)$, provided that $\theta(x) \geqslant r_0$. Next, for every $\delta > 0$ one can find an open set H containing all non-Ahlfors balls (that is, all balls with the property $\mu(B) > Mr^n$ for some constant $M = M(\delta)$) with $\mu(H) \leqslant \delta$ and such that $|S_\varepsilon\mu(x)| \leqslant C(\delta)$ for every $\varepsilon > \text{dist}\{x, \mathbb{R}^d \setminus H\}$, for all $x \in \mathbb{R}^d$. According to the above, this, in particular,

implies that $|S_\theta \mu(x)| \leqslant C'(\delta)$ for every Lipschitz function $\theta(x) \geqslant \mathrm{dist}\{x, \mathbb{R}^d \setminus H\}$ with the Lipschitz constant 1, for all $x \in \mathbb{R}^d$. We claim that for every such θ

$$\left| S_\theta \mu(x) \right| \leqslant C \quad \text{for every } x \in \mathbb{R}^d \quad \Longrightarrow \quad S_\theta : L^2(\mu) \to L^2(\mu). \tag{25}$$

Indeed, let $\overline{S}_\theta$ be the mapping from $f : \mathbb{R}^d \to L^2(\mathrm{d}t)$ to

$$\overline{S}_\theta f(x, t) := \frac{1}{\sqrt{t}} \int\limits_{t \leqslant |x-y| \leqslant 2t} k_\theta(x, y) f(y, t) \, \mathrm{d}\mu(y). \tag{26}$$

Then one can employ the non-homogeneous $T1$ theorem for vector-valued kernels (*Tb* Theorem 4 in [3]), based, in turn, on the earlier developments in [7]. We omit in this Note the detailed verification of the conditions of the theorem. The most important part is that the bound $S_\theta \mu(x) \leqslant C$, $x \in \mathbb{R}^d$, amounts to $\overline{S}_\theta(\mathbf{1}) \in L^\infty(\mu, L^2(\mathrm{d}t))$, where $\mathbf{1}$ is the identity on $\mathbb{R}^d \times (0, \infty)$. This, in a sense, plays a role of the $T1$ condition, and allows to conclude that $\overline{S}_\theta$ is a bounded operator on $L^2(\mu, L^2(\mathrm{d}t))$, so that in particular, S_θ is bounded in $L^2(\mu)$ and (25) holds.

Finally, take $\theta(x) = \mathrm{dist}\{x, \mathbb{R}^d \setminus H\}$ and observe that S_θ coincides with S on $\mathbb{R}^d \setminus H$, so that (25) implies that S is bounded on $L^2(\mu|_{\mathbb{C} \setminus H})$, as desired. $\square$

Finally, it remains to prove Proposition 3.1. This is the most technical part of the argument, which mimics the construction of Lipschitz graphs by Léger in [4]. We shall present the detailed argument in a full-size paper.

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