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COMPTES RENDUS MATHEMATIQUE

Complex Analysis

Analytic continuation of holomorphic mappings

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Abstract

Let D be a domain in $\mathbb{C}^n$, $n > 1$, and $f : D \to \mathbb{C}^n$ be a holomorphic map. Let $U \subset \mathbb{C}^n$ be an open set such that $M := \partial D \cap U$ is in U a relatively closed, connected, smooth real-analytic hypersurface of finite type (in the sense of D'Angelo). Suppose that the cluster set $cl_f(M)$ is contained in a closed, smooth real-algebraic hypersurface $M' \subset U'$ of finite type, where U' is an open set in $\mathbb{C}^n$. It is shown that if f extends continuously to some open piece of M, then it extends holomorphically to a neighborhood of each point of M. Note that here the compactness of M' is not required. ***To cite this article: B. Ayed, N. Ourimi, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Continuation analytique d'applications holomorphes. Soient D un domaine de $\mathbb{C}^n$, $n > 1$, et $f : D \to \mathbb{C}^n$ une application holomorphe. Soit $U \subset \mathbb{C}^n$ un ouvert tel que $M := \partial D \cap U$ est une hypersurface relativement fermée dans U, connexe, lisse, analytique réelle et de type fini (au sens de D'Angelo). Supposons que l'ensemble des points limites $cl_f(M)$ est contenu dans une hypersurface, fermée, lisse, algébrique réelle $M' \subset U'$ de type fini, où U' est un ouvert de $\mathbb{C}^n$. Nous montrons que si f se prolonge continûment sur une partie ouverte de M, alors elle se prolonge holomorphiquement au voisinage de chaque point de M. Notons qu'ici la compacité de M' n'est pas exigée. ***Pour citer cet article : B. Ayed, N. Ourimi, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Version française abrégée

R. Shafikov [9] a montré qu'une application holomorphe propre $f : D \to D'$ entre domaines bornés de $\mathbb{C}^n$, $n > 1$, D est à bord analytique réel et D' est à bord algébrique réel, se prolonge holomorphiquement dans un voisinage de $\bar{D}$. Le but principal de cette Note est d'étudier une version locale de ce résultat. On montre le théorème suivant :

Théorème 0.1. *Soient D un domaine de $\mathbb{C}^n$, $n > 1$, et $f : D \to \mathbb{C}^n$ une application holomorphe. Soit $U \subset \mathbb{C}^n$ un ouvert tel que $M := \partial D \cap U$ est une hypersurface relativement fermée dans U, connexe, lisse, analytique réelle et de type*

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fini (*au sens de D'Angelo*). *Supposons que l'ensemble des points limites $cl_f(M)$ est contenu dans une hypersurface, fermée, lisse, algébrique réelle $M' \subset U'$ de type fini, où U' est un ouvert de $\mathbb{C}^n$. Si f se prolonge continûment sur une partie ouverte de M, alors elle se prolonge holomorphiquement au voisinage de chaque point de M.*

La preuve est basée sur la propagation de l'analycité des applications holomorphes à travers les variétés de Segre et sur la construction d'une famille d'ellipsoïdes utilisée dans [6]. D'abord, nous montrons la proposition suivante :

Proposition 0.2. *Soient $M \subset U \subset \mathbb{C}^n$ et $M' \subset U' \subset \mathbb{C}^n$, $n > 1$, deux hypersurfaces de $\mathbb{C}^n$, $n > 1$, fermées respectivement dans U et U', et de type fini. Supposons que M est lisse, analytique réelle et M' est lisse, algébrique réelle. Supposons aussi que $\Gamma \subset M$ est un ouvert connexe, $\partial\Gamma \cap M$ est une sous-variété générique et $f : \Gamma \to M'$ est une application CR continue. Soit $p \in \partial\Gamma = \bar{\Gamma}\backslash\Gamma$. Alors il existe un voisinage $U_p \ni p$ dans $\mathbb{C}^n$ tel que le graphe de f se prolonge comme un ensemble analytique sur $U_p \times \mathbb{C}^n$.*

Démonstration du Théorème 0.1. Elle est basée sur la Proposition 0.2 et le Théorème 1.1 de [2]. Soit M_{h} l'ensemble des points $z \in M$, où f se prolonge holomorphiquement au voisinage de z. D'après [4] l'ensemble M_{h} est non-vide. Pour montrer que $M_{\mathrm{h}} = M$, il suffit de montrer que M_{h} est fermé dans M (puisque par définition M_{h} est un ouvert). Par l'absurde, supposons que $\bar{M}_{\mathrm{h}} \neq M_{\mathrm{h}}$ et soit $q \in \partial M_{\mathrm{h}} = \bar{M}_{\mathrm{h}}\backslash M_{\mathrm{h}}$. L'hypersurface M étant globalement minimale, il existe une courbe CR lisse γ de M passant par q et intersectant M_{h} (γ est dite CR si le vecteur tangent à γ en tout point est contenu dans l'espace tangent complexe T^cM). La preuve est basée sur la construction d'une famille d'ellipsoïdes utilisée par Merker et Porten dans [6]. Dans un voisinage de q la courbe γ peut être décrite comme une partie d'une courbe intégrale d'un certain champ de vecteur L, CR (i.e., section de T^cM) et sans zéro. En intégrant L, nous obtenons un système de coordonnées lisses $(t,s) \in \mathbb{R} \times \mathbb{R}^{2n-2}$ sur M tel que pour s_0 fixé les segments (t, s_0) sont contenus dans les trajectoires de L. Sans perdre de généralité, supposons que $0 \in \gamma \cap M_{\mathrm{h}}$ et 0 est très proche de q. Pour $\epsilon > 0$ et $\tau > 0$ nous définissons la famille d'ellipsoïdes sur M de centre 0 par : $E_\tau = \{(t,s)\colon |t|^2/\tau + |s|^2 < \epsilon\}$. Observons que pour chaque τ l'ensemble ∂E_τ est transverse aux trajectoires de L en dehors de l'ensemble $\Upsilon = \{(0,s)\colon |s|^2 = \epsilon\}$. Par conséquent, ∂E_τ est générique en tout point excepté les points de Υ. Notons que Υ est contenu dans M_{h}. Fixons $\epsilon > 0$ assez petit tel que pour un certain $\tau_0 > 0$ l'ellipsoïde E_{τ_0} est relativement compact et contenu dans M_{h}. Soit τ_1 le plus petit nombre réel tel que f ne se prolonge pas holomorphiquement en un certain point $b \in \partial E_{\tau_1}$. Notons que $\tau_1 > \tau_0$ et que le point b peut être a priori différent du point q. La contradiction consiste à montrer que f se prolonge holomorphiquement dans un voisinage de b. Près du point b, l'ensemble ∂E_{τ_1} est une sous-variété lisse générique de M, puisque les points non-génériques de ∂E_{τ_1} sont contenus dans Υ, qui est une partie de M_{h}. D'après la Proposition 0.2, il existe un voisinage $U_b \ni b$ dans $\mathbb{C}^n$ tel que le graphe de f se prolonge comme un ensemble analytique sur $U_b \times \mathbb{C}^n$. D'après [2], f se prolonge holomorphiquement dans un voisinage de b. Cette contradiction montre que $M_{\mathrm{h}} = M$ et achève la démonstration du théorème. □

1. Introduction

It was proved in [9] that a proper holomorphic mapping $f : D \to D'$ from a bounded domain in $\mathbb{C}^n$ with a smooth real-analytic boundary onto a bounded domain in $\mathbb{C}^n$ ($n > 1$) with a smooth real-algebraic boundary extends holomorphically to a neighborhood of $\bar{D}$. The proof is based on the idea of analytic continuation of holomorphic mappings along Segre varieties. The main purpose of this Note is to prove a local version of this result. More precisely we prove the following:

Theorem 1.1. *Let D be a domain in $\mathbb{C}^n$, $n > 1$, and $f : D \to \mathbb{C}^n$ be a holomorphic map. Let $U \subset \mathbb{C}^n$ be an open set such that $M := \partial D \cap U$ is in U a relatively closed, connected, smooth real-analytic hypersurface of finite type* (*in the sense of D'Angelo*). *Suppose that the cluster set $cl_f(M)$ is contained in a closed, smooth real-algebraic hypersurface $M' \subset U'$ of finite type, where U' is an open set in $\mathbb{C}^n$. If f extends continuously to some open piece of M, then it extends holomorphically to a neighborhood of each point of M.*

Note that f is not assumed to be proper. A related result was proved in [11] for proper holomorphic maps between bounded domains in $\mathbb{C}^n$ of different dimensions (in [11] the holomorphic extension was proved only on a dense subset of the boundary). Our result (even the holomorphic extension on a dense subset of the boundary) cannot directly derived from [11], since we do not require compactness of M'. Theorem 1.1 is already known if $M \subset W \subset \mathbb{C}^n$ and $M' \subset W' \subset \mathbb{C}^n$ are closed, connected, smooth real-analytic hypersurfaces of finite type in some open set W respectively W' in $\mathbb{C}^n$ and $f : M \to M'$ is a continuous CR-map (see [4], and see also [5] for related results). For $n = 2$, the result was proved in [10] in the real-analytic case for proper holomorphic mappings, but without assuming that the mapping is known to be continuous. If we assume that f is a germ of a holomorphic mapping from M to M' (not necessarily holomorphic in the whole of D), then f may not extend holomorphically to certain points of M, as the following example (which appeared in [8]) shows:

Example 1. Let $M' = \{(z_1', z_2') \in \mathbb{C}^2\colon\ |z_1'|^2 + |z_2'|^4 = 1\}$. Then $f(z_1, z_2) = (z_1, \sqrt{z_2})$ maps $\partial\mathbb{B}^2$ to M', but f cannot be extended as a holomorphic mapping to a neighborhood of $(1, 0)$.

Note that in this example f is not holomorphic in the whole of $\mathbb{B}^2$ ($\mathbb{B}^2$ denotes the unit ball in $\mathbb{C}^2$). However, if f is a germ of a holomorphic mapping from M to M' and in addition, M' is compact and strictly pseudoconvex, then f extends holomorphically along any path on M (see [8]).

In the proof of [9] the author uses mainly the fact that ∂D^- (the set of points where the Levi-form of the defining function of D has at least one negative eigenvalue) is non-empty. In Theorem 1.1 the situation is different, since a priori M and M' may be respectively pseudoconvex and non-pseudoconvex and this is the main reason why the local version of this result can not directly derived from [9]. However, if M is not pseudoconvex and $f : D \to D'$ is a holomorphic mapping with $cl_f(M) \subset M'$, then we can repeat the argument of [9] with Proposition 2.1 (see Section 2) and the result of [2] to show that f extends holomorphically past M. Obviously the result of [9] cannot be extended to unbounded domains, since the cluster set of a boundary point may contain only the point at infinity as the following example shows:

Example 2. Let $\Omega = \{(z_0, z) \in \mathbb{C} \times \mathbb{C}^n\colon\ 2Re(z_0) + |z|^2 < 0\}$ be the unbounded representation of the unit ball. We define

$$f : (z_0, z) \mapsto \left(\frac{1}{z_0}, \frac{z}{z_0}\right).$$

Clearly, f is an analytic automorphism of Ω. It extends holomorphically everywhere except at the origin. Note that here $cl_f(0) = \{\infty\}$.

2. Extension as an analytic set

The propagation of analyticity of holomorphic mappings along Segre varieties is given in the following proposition:

Proposition 2.1. *Let M be a smooth real-analytic, essentially finite hypersurface in $\mathbb{C}^n$, $n > 1$, and let $U_1 \Subset U_2$ be a standard pair of neighborhoods of a point $a \in M$. Let $f : U_a \to f(U_a) \subset \mathbb{C}^n$ be a biholomorphic mapping such that $f(U_a \cap M) \subset M'$, where U_a is an arbitrary small neighborhood of a, $U_a \subset U_1$ and $M' \subset \mathbb{C}^n$ is a smooth real-algebraic essentially finite hypersurface. Then there exist V a neighborhood of $Q_a \cap U_1$ and an analytic set $\Lambda \subset V$, $\dim_{\mathbb{C}} \Lambda \leqslant n - 1$ such that for any $b \in (Q_a \cap U_1)\backslash\Lambda$, the graph of f extends as an analytic set to $W \times \mathbb{C}^n$, where W is a neighborhood of the connected component of $Q_b \cap M \cap U_1$ containing a.*

Related results were proved in [9] and [8] when M' is compact. Recall that M is called essentially finite at p if there exists a sufficiently small neighborhood $U_p \ni p$ such that $I_p = \{z \in U_p\colon\ Q_z = Q_p\} = \{p\}$. Note that any real-analytic hypersurface of finite type is essentially finite. We refer the reader to [3] (with references included) for definitions and details on Segre varieties. For the definition of the extension of the graph of a mapping as an analytic set, see for examples [7] or [2].

Proof of Proposition 2.1. The proof is in two steps. First, to construct a correspondence in a neighborhood of $(Q_a \cap U_1)\backslash\Lambda$ and second to extend the graph of f to an analytic set as in Proposition 2.1.

Step 1. We shrink U_a and choose V a neighborhood of $Q_a \cap U_1$ in such a way that for any $w \in V$, the set $Q_w \cap U_a$ is connected. Observe that if V is small enough then $Q_w \cap U_a \neq \emptyset$ for any $w \in V$, as $w \in Q_a$ implies $a \in Q_w$. We define

$$X = \{(w, w') \in V \times \mathbb{C}^n \colon\ f(Q_w \cap U_a) \subset Q'_{w'}\}.$$

We would like to have $Q_w \cap U_a$ connected for any $w \in V$ to avoid ambiguity in the condition $f(Q_w \cap U_a) \subset Q'_{w'}$, since different components of $Q_w \cap U_a$ could be mapped a priori to different Segre varieties. According to [8], X is an analytic set in $V \times \mathbb{C}^n$. By the invariance property of Segre varieties, X extends the graph of $f_{|V\cap U_a}$ (the restriction of f to $V \cap U_a$). From the algebraicity of M' the set X extends to an analytic set in $V \times \mathbb{P}^n$ (for details see [8], Proposition 3.1). This extension will be denoted by $\bar{X}$. Let $\pi : \bar{X} \to V$ and $\pi' : \bar{X} \to \mathbb{P}^n$ be the natural projections. We define the following sets: $\Lambda_1 = \pi(\pi'^{-1}(H_0))$ and $\Lambda_2 = \pi\{(w, w') \in \bar{X}\colon\ \dim \pi^{-1}(w) \geqslant 1\}$, where $H_0 \subset \mathbb{P}^n$ is the hyperplane at infinity. Note that Λ_1 is a complex manifold of dimension at most $n-1$ in $\mathbb{C}^n$ and Λ_2 is a complex analytic set of dimension at most $n-2$ (see Proposition 3.3 in [8]). The projection π is proper, since $\mathbb{P}^n$ is compact. Let $\Lambda = \Lambda_1 \cup \Lambda_2$. Now, it is clear that the restriction $(\pi' \circ \pi^{-1})_{|V\setminus\Lambda}$ is a holomorphic correspondence extending $f_{|V\cap U_a}$.

Step 2. Consider the restriction of the extended correspondence to some neighborhood U_b of b, $b \in (Q_a \cap U_1)\setminus\Lambda$ and $U_b \subset V$. Let $F : U_b \to \mathbb{C}^n$ be the corresponding multiply-valued mapping and W be a neighborhood of the connected component of $Q_b \cap M \cap U_1$ that contains a (we choose W and we shrink U_b so that for all $w \in W$, $Q_w \cap U_b$ is non-empty and connected). We denote by S_F the branch locus of F (i.e., $z \in S_F$ if the coordinate projection π is not locally biholomorphic near $\pi^{-1}(z)$) and let $\Sigma = \{w \in W\colon\ Q_w \cap U_b \subset S_F\}$. Consider the set

$$X^* = \{(w, w') \in (W\setminus\Sigma) \times \mathbb{C}^n\colon\ F(Q_w \cap U_b) \subset Q'_{w'}\}.$$

Claim.

i) *X^* is not empty*;
ii) *X^* is locally a complex analytic set*;
iii) *X^* is closed in $(W\setminus\Sigma) \times \mathbb{C}^n$*;
iv) *$\Sigma \times \mathbb{C}^n$ is a removable singularity for X^*.*

Proof of the claim. i) By the invariance property of Segre varieties (see for instance [3]) $X^* \neq \emptyset$.

ii) Let $(w, w') \in X^*$. Consider an open simply connected set $\Omega \subset U_b\setminus S_F$ such that $Q_w \cap \Omega \neq \emptyset$. The branches of F are globally defined in Ω. Since $Q_w \cap U_b$ is connected, the inclusion $F(Q_w \cap U_b) \subset Q'_{w'}$ is equivalent to: $F^j(Q_w \cap \Omega) \subset Q'_{w'}$, $j = 1, \dots, m$, where the F^j denote the branches of F in Ω. Let $P'(w', \bar{w}')$ be a defining polynomial of M'. The inclusion $F^j(Q_w \cap \Omega) \subset Q'_{w'}$, $j = 1, \dots, m$ can be expressed as

$$P'(F^j(z), \bar{w}') = 0 \quad \text{for any } z \in Q_w \cap \Omega,\ j = 1, \dots, m.$$

Since we can choose Ω in the form $\Omega = \Omega_1 \times {}'\Omega \subset \mathbb{C} \times \mathbb{C}^{n-1}$ such that $Q_w = \{(h('z, \bar{w}), 'z),\ 'z \in {}'\Omega\}$ ($h('z, \bar{w})$ is holomorphic in $'z$ and antiholomorphic in w), we obtain

$$P'\big(F^j\big(h('z, \bar{w}), 'z\big), \bar{w}'\big) = 0, \quad \text{for any } 'z \in {}'\Omega. \tag{*}$$

Thus, X^* is defined by an infinite system of holomorphic equations in $(\bar{w}, \bar{w}')$. By the Noetherian property of the ring of holomorphic functions, we can choose finitely many points $'z^1, \dots, 'z^m$ so that (*) can be written as a finite system

$$\sum_{|J|\leqslant d'} \alpha_J^k(w) w'^J = 0,$$

where $k = 1, \dots, m$, d' is the degree of P' in w' and α_J^k are holomorphic functions in w. Thus, X^* is locally a complex analytic set in $(W\setminus\Sigma) \times \mathbb{C}^n$.

iii) The set X^* is closed in $(W\setminus\Sigma) \times \mathbb{C}^n$. Indeed; let (w_j, w'_j) be a sequence in X^* that converges to $(w_o, w'_o) \in (W\setminus\Sigma) \times \mathbb{C}^n$, as $j \to \infty$. Since $Q_{w_j} \to Q_{w_o}$ and $Q'_{w'_j} \to Q'_{w'_o}$, from the inclusion $F(Q_{w_j} \cap U_b) \subset Q'_{w'_j}$ we obtain $F(Q_{w_o} \cap U_b) \subset Q'_{w'_o}$, which implies that $(w_o, w'_o) \in X^*$ and thus, X^* is a closed set.

iv) Now, let us show that $\Sigma \times \mathbb{C}^n$ is a removable singularity for X^*. Let $t \in \Sigma$. It follows that $\bar{X}^* \cap (\{t\} \times \mathbb{C}^n) \subset \{t\} \times \{z'\colon\ F(Q_t \cap U_b) \subset Q'_{z'}\}$. If $w' \in F(Q_t) \subset Q'_{z'}$, then $z' \in Q'_{w'}$. Since $\dim_{\mathbb{C}} Q'_{w'} = n-1$, then $\{z'\colon\ F(Q_t \cap U_b) \subset Q'_{z'}\}$ has dimension at most $2n-2$ and $\bar{X}^* \cap (\Sigma \times \mathbb{C}^n)$ has $2n$-dimensional measure zero. Now, Bishop's theorem can be applied to conclude that $\Sigma \times \mathbb{C}^n$ is a removable singularity for X^* and consequently $\bar{X}^*$ is an analytic set in $W \times \mathbb{C}^n$. This finishes the proof of the claim. □

Now, according to [9], Lemma 1, $\bar{X}^* \cap [(W \cap U_a \cap V) \times \mathbb{C}^n] = X \cap [(W \cap U_a \cap V) \times \mathbb{C}^n]$. This shows that $\bar{X}^*$ contains the graph of $f_{|W \cap U_a \cap V}$. Hence, $\bar{X}^*$ is the desired analytic set. □

By using Proposition 2.1 and Proposition 4.1 in [8] we establish the following:

Proposition 2.2. *Let $M \subset U \subset \mathbb{C}^n$ and $M' \subset U' \subset \mathbb{C}^n$, $n > 1$, be closed hypersurfaces of finite type in some open set U respectively U' in $\mathbb{C}^n$. Assume that M is smooth real-analytic and M' is smooth real-algebraic. Further, suppose that $\Gamma \subset M$ is a connected open set and $\partial\Gamma \cap M$ is a smooth generic submanifold. Let $f : \Gamma \to M'$ be a continuous CR-map and p be a point in $\partial\Gamma = \bar{\Gamma}\backslash\Gamma$. Then there exists a neighborhood $U_p \ni p$ in $\mathbb{C}^n$ such that the graph of f extends as an analytic set to $U_p \times \mathbb{C}^n$.*

Proof of Proposition 2.2. According to [4], f extends holomorphically to a neighborhood of Γ. Without loss of generality we may assume that f is non-constant. It follows from a result of Baouendi and Rothschild (see [1]) that J_f; the Jacobian of f, is not identically equal to zero. Let $U_1 \Subset U_2$ be a standard pair of neighborhoods of p. According to [8], there exists an open subset $\omega \subset Q_p \cap U_1$ such that for all $b \in \omega$, $Q_b \cap \Gamma \neq \emptyset$. Moreover, there exists a non-constant closed path $\gamma \subset (Q_b \cap \Gamma) \cup \{p\}$ with the end point at p. In view of [8] (Remark 2 following Proposition 4.1) Q_b intersects Γ transversally. Since M is of finite type, then in a small neighborhood U_b of b the set $\{z \in U_b\colon\ J_f|_{Q_z \cap \Gamma} \equiv 0\}$ is finite. Hence, by moving b if necessary, we may assume that $J_f|_{Q_b \cap \Gamma}$ is not identically zero. Let $\Lambda = \Lambda_1 \cup \Lambda_2$ be the set defined in Proposition 2.1. Recall that $\dim_{\mathbb{C}} \Lambda_2 \leqslant n-2$ and $\dim_{\mathbb{C}} \Lambda_1 \leqslant n-1$. We may choose the triplet (b, γ, a) such that $b \in \omega\backslash\Lambda_2$, $a \in \gamma \subset Q_b \cap \Gamma$ so close to p that, possibly after a small perturbation U_1, U_2 will also be a standard pair of neighborhoods of a and $J_f(a) \neq 0$. Notice that $\dim_{\mathbb{C}}(Q_a \cap \omega) = n-2$ and $b \in Q_a \cap \omega$. Since $b \in Q_a \cap \omega$, then $Q_a \cap \omega \not\subset \Lambda_2$. Also, we may exclude the case when $Q_a \cap \omega \subset \Lambda_1$. Indeed; if $Q_a \cap \omega \subset \Lambda_1$, we can perform a linear-fractional transformation in $\mathbb{P}^n$ such H_0 is mapped onto another complex hyperplane $H \subset \mathbb{P}^n$ with $H \cap M' = \emptyset$. Hence, we may assume that $(Q_a \cap \omega) \not\subset \Lambda$ and if necessary, we may replace b by another point in $(Q_a \cap \omega)\backslash\Lambda$. Now, the result follows from Proposition 2.1. □

3. Proof of Theorem 1.1

Let $M_{\mathrm{h}} = \{z \in M\colon\ f$ extends holomorphically to a neighborhood of $z\}$. According to [4], M_{h} is not empty. It is open by construction. To prove Theorem 1.1, it suffices to show that M_{h} is closed in M. We argue by contradiction. Assume that $\bar{M}_{\mathrm{h}} \neq M_{\mathrm{h}}$ and let $q \in \partial M_{\mathrm{h}} = \bar{M}_{\mathrm{h}}\backslash M_{\mathrm{h}}$. We follow the idea in [11]. Since M is globally minimal, there exists a CR-curve γ (i.e., the tangent vector to γ at any point is contained in the complex tangent to M) passing through q and entering M_{h}. We will use the construction of a family of ellipsoids used by Merker and Porten [6]. After shortening γ we may assume γ is a smoothly embedded segment. Then γ can be described as a part of an integral curve of some non-vanishing smooth CR vector field L (i.e., section of T^cM) near q. Integrating L, we obtain a smooth coordinate system $(t, s) \in \mathbb{R} \times \mathbb{R}^{2n-2}$ on M such that for any fixed s_0 the segments (t, s_0) are contained in the trajectories of L. After a translation, we may assume that $0 \in \gamma \cap M_{\mathrm{h}}$ and 0 is close to q. For $\epsilon > 0$ and $\tau > 0$ we define the family of ellipsoids on M centered at 0 by

$$E_\tau = \bigl\{(t, s)\colon\ |t|^2/\tau + |s|^2 < \epsilon\bigr\}.$$

Observe that every ∂E_τ is transverse to the trajectories of L out off the set $\Upsilon = \{(0, s)\colon\ |s|^2 = \epsilon\}$. So, ∂E_τ is generic at every point except the set Υ. Note that Υ is contained in M_{h}. Fix $\epsilon > 0$ so small such that for some $\tau_0 > 0$ the ellipsoid E_{τ_0} is compactly contained in M_{h}. Let τ_1 be the smallest positive number such that f does not extend holomorphically to some point $b \in \partial E_{\tau_1}$. Note that $\tau_1 > \tau_0$ and b may be different from q. The contradiction is to show that f extends holomorphically to a neighborhood of b. Near b, ∂E_{τ_1} is a smooth generic manifold of M, since the non-generic points of ∂E_{τ_1} are contained in Υ, which is contained in M_{h}. By Proposition 2.2, there exists a neighborhood $U_b \ni b$ in $\mathbb{C}^n$ such that the graph of f extends as an analytic set to $U_b \times \mathbb{C}^n$. According to [2], f extends holomorphically to a neighborhood of b. This contradiction proves that $M_{\mathrm{h}} = M$ and finishes the proof of the theorem.

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