Available online at www.sciencedirect.com
ScienceDirect
C. R. Acad. Sci. Paris, Ser. I 347 (2009) 1099–1103
COMPTES RENDUS MATHEMATIQUE

Mathematical Problems in Mechanics/Calculus of Variations

Decomposition of shell deformations – Asymptotic behavior of the Green–St Venant strain tensor

Dominique Blanchard [a], Georges Griso [b]

[a] *Université de Rouen, UMR 6085, laboratoire Raphaël-Salem, 76801 St Etienne du Rouvray cedex, France*
[b] *Laboratoire d'analyse numérique, université P. et M. Curie, case courrier 187, 75252 Paris cedex 05, France*

Received 3 June 2009; accepted 30 June 2009

Available online 25 July 2009

Presented by Philippe G. Ciarlet

Abstract

This Note deals with a new method, based on a decomposition of the deformations, to study thin shells. In particular, we give the asymptotic behavior of the Green–St Venant's strain tensor. ***To cite this article: D. Blanchard, G. Griso, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***
© 2009 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

Résumé

Décomposition des déformations des coques – Comportement asymptotique du tenseur des déformations de Green–St Venant. Dans cette Note nous présentons une nouvelle méthode, basée sur une décomposition des déformations, pour l'étude des coques minces. En particulier, nous donnons le comportement asymptotique du tenseur de Green–St Venant. ***Pour citer cet article : D. Blanchard, G. Griso, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***
© 2009 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

Version française abrégée

Nous proposons une nouvelle méthode d'analyse du comportement des grandes déformations de coques minces. Cette méthode s'appuie sur une décomposition des déformations qui prend en compte la faible épaisseur du domaine. Cette démarche a été introduite pour les petites déformations dans [8] et [9] et développée dans [1] pour des poutres élastiques en grandes déformations. Une déformation v d'une coque de surface moyenne S, de normale $\mathbf{n}$ et d'épaisseur 2δ s'écrit

$$v = \mathcal{V} + s_3\mathbf{R}\mathbf{n} + \bar{v}$$

où $\mathcal{V}$ et $\mathbf{R}$ sont définis sur S. Le champ $\mathcal{V}$ représente la déformation de la surface moyenne S, le champ de matrices $\mathbf{R}$ représente la rotation des fibres et $\bar{v}$ le gauchissement des fibres. Dans le cadre de la mécanique des milieux continus, la décomposition ci-dessus apparaît comme une écriture naturelle de la déformation v pour δ petit. Dans le Théo-

E-mail addresses: dominique.blanchard@univ-rouen.fr (D. Blanchard), griso@ann.jussieu.fr (G. Griso).

1631-073X/$ – see front matter © 2009 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.
doi:10.1016/j.crma.2009.06.018

rème 1 nous donnons des estimations de $\mathcal{V}$, $\mathbf{R}$ et $\bar{v}$ à l'aide de l'« *énergie de déformation* » $\| \operatorname{dist}(\nabla_x v, SO(3))\|_{L^2}$. La démonstration de ce Théorème s'appuie sur le Théorème de rigidité géométrique de [5] (avec les dépendances géométriques des constantes données dans [1]).

Une première application importante de cette technique est l'obtention d'inégalités de type Korn pour les coques minces (voir [4] pour de telles inégalités pour les domaines 3d). Nous obtenons aussi une estimation du tenseur linéarisé des déformations (Théorème 3). Ces inégalités et estimations permettent de dégager des cas critiques pour l'ordre en δ de l'« *énergie de déformation* ».

Une seconde application originale est le calcul de la limite du tenseur des déformations de Green–St Venant en fonction des limites des termes de notre décomposition (tous ces élements ayant un sens mécanique). Nous présentons ces résultats dans deux cas critiques : $\| \operatorname{dist}(\nabla_x v, SO(3))\|_{L^2}$ d'ordre $\delta^{3/2}$ (Lemme 4 et Théorème 5) et d'ordre $\delta^{5/2}$ (Lemme 6 et Théorème 7). Dans le premier cas la déformation limite est en flexion pure mais le tenseur de Green–St Venant limite fait apparaître un terme qui mesure le défaut entre la déformation moyenne et une déformation de flexion pure. Dans le deuxième cas les déplacements limites des fibres de la coque sont rigides. Le défaut mentionné ci-dessus s'exprime en fonction du déplacement de flexion et d'un déplacement extensionnel qui est l'analogue d'un déplacement de membrane dans une plaque. Les résultats présentés ci-dessus permettent aussi d'obtenir des modèles limites de coques élastiques, en flexion pure ou de type Von Kármán, par exemple par Γ-convergence.

1. Introduction

We introduce a decomposition technique for the large deformations of a thin shell which takes into account the fact that the thickness 2δ of such a domain is small. This technique has been developed in the framework of linearized strain for thin structures in e.g. [8] and [9]. We have already developed the same method to study nonlinear elastic rods (see [1]). The main tool is to derive estimates on the terms of the decomposition using the "*strain energy*" $\| \operatorname{dist}(\nabla_x v, SO(3))\|_{L^2}$ (see Theorem 1). The first important application is the obtention of a nonlinear Korn's inequality for shells (for 3d bodies such inequalities are established in [4]) and of an estimate on the linearized strain tensor. From all these estimates arise critical cases for the level of the "*strain energy*". Among these critical cases, we distinguish two main examples for which we investigate the limit of the Green–St Venant's strain tensor. Another application of our decomposition technique could also be the derivation of nonlinear bending models or Von Kármán's type models for elastic shell (for the derivation of elastic plates and shells models see e.g. [2,6,7] and [10]).

The detailed proofs will be presented in forthcoming papers.

2. The geometry and notations

Let ω be a bounded domain in $\mathbb{R}^2$ with a Lipschitzian boundary. The mid-surface S of the shell is defined as $\phi(\overline{\omega})$ where ϕ is a $\mathcal{C}^2$-injective mapping from $\overline{\omega}$ into $\mathbb{R}^3$. We set $\mathbf{t}_1 = \frac{\partial \phi}{\partial s_1}$, $\mathbf{t}_2 = \frac{\partial \phi}{\partial s_2}$ and $\mathbf{n} = \frac{\mathbf{t}_1 \wedge \mathbf{t}_2}{\|\mathbf{t}_1 \wedge \mathbf{t}_2\|_2}$ and we assume that the tangential vectors $\mathbf{t}_1$ and $\mathbf{t}_2$ are linearly independent at each point of $\overline{\omega}$. We set $\Omega_\delta = \omega \times]-\delta, \delta[$. For δ small enough, the shell $\mathcal{Q}_\delta$ is defined as $\mathcal{Q}_\delta = \Phi(\Omega_\delta)$ where Φ is the map $(s_1, s_2, s_3) \mapsto x = \phi(s_1, s_2) + s_3 \mathbf{n}(s_1, s_2)$.

We make the convention that a function $v(x)$ for $x \in \mathcal{Q}_\delta$ is still denoted by $v(s)$ for $s \in \Omega_\delta$ $(x = \Phi(s))$.

3. Decomposition of a deformation

Let v be a deformation in $(H^1(\mathcal{Q}_\delta))^3$. We decompose v as the sum of a particular deformation and a residual part. We establish estimates on the terms of this decomposition with respect to the "*strain energy*" $\| \operatorname{dist}(\nabla_x v, SO(3))\|_{L^2(\mathcal{Q}_\delta)}$.

Theorem 1. *There exists a constant $C(S)$ such that for all $v \in (H^1(\mathcal{Q}_\delta))^3$ satisfying*

$$\left\| \operatorname{dist}\big(\nabla_x v, SO(3)\big)\right\|_{L^2(\mathcal{Q}_\delta)} \leqslant C(S)\delta^{3/2}, \tag{1}$$

there exist $\mathcal{V} \in (H^1(\omega))^3$, $\mathbf{R} \in (H^1(\omega))^{3\times 3}$ such that $\mathbf{R}(s_1, s_2) \in SO(3)$ for almost all $(s_1, s_2) \in \omega$ and a deformation $\bar{v} \in (H^1(\Omega_\delta))^3$ such that

$$v(s) = \mathcal{V}(s_1, s_2) + s_3 \mathbf{R}(s_1, s_2)\mathbf{n}(s_1, s_2) + \bar{v}(s) \quad \textit{for almost all } s \in \Omega_\delta, \tag{2}$$

and the following estimates hold, where the constant C does not depend on δ:

$$\begin{cases} \frac{1}{\delta}\|\bar{v}\|_{(L^2(\Omega_\delta))^3} + \|\nabla_s \bar{v}\|_{(L^2(\Omega_\delta))^9} \leqslant C\|\operatorname{dist}(\nabla_x v, SO(3))\|_{L^2(\mathcal{Q}_\delta)}, \\ \delta\|\frac{\partial \mathbf{R}}{\partial s_\alpha}\|_{(L^2(\omega))^9} + \|\frac{\partial \mathcal{V}}{\partial s_\alpha} - \mathbf{R}\mathbf{t}_\alpha\|_{(L^2(\omega))^3} \leqslant \frac{C}{\delta^{1/2}}\|\operatorname{dist}(\nabla_x v, SO(3))\|_{L^2(\mathcal{Q}_\delta)}, \\ \|\nabla_x v - \mathbf{R}\|_{(L^2(\Omega_\delta))^9} \leqslant C\|\operatorname{dist}(\nabla_x v, SO(3))\|_{L^2(\mathcal{Q}_\delta)}. \end{cases} \tag{3}$$

In the above decomposition, the field $\mathcal{V}$ stands for the mid-surface deformation, while $\mathbf{R}$ stands for the rotation of the normal fibers and $\bar{v}$ for the warping of these fibers. Assumption (1) allows to construct a rotation field $\mathbf{R}$ such that (2) holds. If (1) does not hold, we can still obtain a decomposition like (2) but with a matrix field $\mathbf{R}$ whose distance to $SO(3)$ is controlled by the "*strain energy*".

Idea of proof. First of all, we extend the deformation v in the neighborhood of the lateral boundary of the shell with a "*strain energy*" of the same order (as developed in [8]). Then, we slice the extended reference domain of thickness 2δ in small cubes of size 2δ and we apply in each cubes the geometric rigidity theorem of [5] to the extended deformation (in the version proved in [1]). This allows to define a continuous path with value in $SO(3)$ on the edges of each square of size 2δ which intersects $\overline{\omega}$. Now, condition (1) allows to construct a H^1-lifting of these paths. The first term $\mathcal{V}$ in (2) is the average of v over the shell fibers. We then proceed as in [1] and [8] in order to obtain all the estimates (3). □

4. Korn's inequality for shells

From now on we consider a clamped condition on a part of the lateral boundary of $\mathcal{Q}_\delta$. Let γ_0 be a connected subset of $\partial\omega$ with positive measure. We set $\Gamma_{0,\delta} = \Phi(\gamma_0 \times]-\delta, \delta[)$.

Let v be a deformation in $(H^1(\mathcal{Q}_\delta))^3$ satisfying $v(x) = x$ on $\Gamma_{0,\delta}$. Then $\mathcal{V}$ and $\mathbf{R}$ can be chosen such that $\mathcal{V} = \phi$ and $\mathbf{R} = \mathbf{I}_3$ (the unit 3×3 matrix) on γ_0 in Theorem 1. This agrees with the mechanical interpretation of $\mathcal{V}$ and $\mathbf{R}$. The following theorem gives a nonlinear Korn's inequality for shells and an estimate for the linearized strain tensor $(\nabla_x v)_S = 1/2((\nabla_x v)^T + \nabla_x v)$. Another Korn's inequality is established in [3].

Theorem 3. *For any deformation* v *in* $\mathbf{U}_\delta = \{v \in (H^1(\mathcal{Q}_\delta))^3 \mid v(x) = x \text{ on } \Gamma_{0,\delta}\}$ *we have* (*the constants do not depend on* δ)

$$\begin{cases} \|v - I_d\|_{(L^2(\mathcal{Q}_\delta))^3} + \|\nabla_x v - \mathbf{I}_3\|_{(L^2(\mathcal{Q}_\delta))^9} \leqslant \frac{C}{\delta}\|\operatorname{dist}(\nabla_x v, SO(3))\|_{L^2(\mathcal{Q}_\delta)}, \\ \|(v - I_d) - (\mathcal{V} - \phi)\|_{(L^2(\mathcal{Q}_\delta))^3} \leqslant C\|\operatorname{dist}(\nabla_x v, SO(3))\|_{L^2(\mathcal{Q}_\delta)}. \end{cases} \tag{4}$$

Moreover we have

$$\left\|(\nabla_x v)_S - \mathbf{I}_3\right\|_{(L^2(\mathcal{Q}_\delta))^9} \leqslant C\left\|\operatorname{dist}\left(\nabla_x v, SO(3)\right)\right\|_{L^2(\mathcal{Q}_\delta)}\left\{1 + \frac{1}{\delta^{5/2}}\left\|\operatorname{dist}\left(\nabla_x v, SO(3)\right)\right\|_{L^2(\mathcal{Q}_\delta)}\right\}. \tag{5}$$

Proof. If v satisfies (1) then the decomposition (2) with estimates (3) and Poincaré's inequality allow to prove (4). For a general v we use the decomposition mentioned after Theorem 1. Estimating $\mathbf{R}^T + \mathbf{R} - 2\mathbf{I}_3$ allows to compare the symmetric part of $\nabla_x v$ and the matrix $\mathbf{I}_3$. □

The inequalities (3) and (5) show that we have two critical cases, when $\|\operatorname{dist}(\nabla_x v, SO(3))\|_{L^2(\mathcal{Q}_\delta)}$ is of order $\delta^{3/2}$ or $\delta^{5/2}$. In particular, the decomposition and Korn's inequality have a useful application for nonlinear elastic shells: as an example for a standard St Venant–Kirchhoff material, they allow to scale the applied forces in order to obtain a desired level of "*strain energy*".

5. Limit of the Green–St Venant's strain tensor for $\|\operatorname{dist}(\nabla_x v, SO(3))\|_{L^2(\mathcal{Q}_\delta)}$ of order $\delta^{3/2}$

The domain Ω_δ is rescaled using the operator

$$(\Pi_\delta w)(s_1, s_2, S_3) = w(s_1, s_2, s_3) \quad \text{for any } s \in \Omega_\delta$$

defined e.g. for $w \in L^2(\Omega_\delta)$ for which $(\Pi_\delta w) \in L^2(\Omega)$.

Let $(v_\delta) \subset \mathbf{U}_\delta$ be a sequence of deformations satisfying (1), and denote by $\mathcal{V}_\delta$, $\mathbf{R}_\delta$ and $\bar{v}_\delta$ the terms of the decomposition of v_δ. As a consequence of (3) we get:

Lemma 4. *There exists a subsequence still indexed by δ such that*

$$\begin{cases} \mathcal{V}_\delta \to \mathcal{V} \text{ strongly in } (H^1(\omega))^3, \\ \mathbf{R}_\delta \rightharpoonup \mathbf{R} \text{ weakly in } (H^1(\omega))^{3\times 3} \text{ and strongly in } (L^2(\omega))^{3\times 3}, \\ \frac{1}{\delta^2}\Pi_\delta \bar{v}_\delta \rightharpoonup \bar{v} \text{ weakly in } (L^2(\omega; H^1(-1,1)))^3, \\ \frac{1}{\delta}\big(\frac{\partial \mathcal{V}_\delta}{\partial s_\alpha} - \mathbf{R}_\delta \mathbf{t}_\alpha\big) \rightharpoonup \mathcal{Z}_\alpha \text{ weakly in } (L^2(\omega))^3, \end{cases} \tag{6}$$

where $\mathbf{R}$ belongs to $SO(3)$ almost everywhere in ω. We also have $\mathcal{V} \in (H^2(\omega))^3$ and $\frac{\partial \mathcal{V}}{\partial s_\alpha} = \mathbf{R}\mathbf{t}_\alpha$. The boundaries conditions $\mathcal{V} = \phi$ and $\mathbf{R} = \mathbf{I}_3$ on γ_0 hold. Moreover, we have

$$\begin{cases} \Pi_\delta v_\delta \to \mathcal{V} \text{ strongly in } (H^1(\Omega))^3, \\ \Pi_\delta(\nabla_x v_\delta) \to \mathbf{R} \text{ strongly in } (L^2(\Omega))^9. \end{cases} \tag{7}$$

Theorem 5. *For the same subsequence as in Lemma 4, we have*

$$\frac{1}{2\delta}\Pi_\delta\big((\nabla_x v_\delta)^T \nabla_x v_\delta - \mathbf{I}_3\big) \rightharpoonup \big(\mathbf{t}_1 \mid \mathbf{t}_2 \mid \mathbf{n}\big)^{-T}\mathbf{E}\big(\mathbf{t}_1 \mid \mathbf{t}_2 \mid \mathbf{n}\big)^{-1} \quad \text{weakly in } \big(L^1(\Omega)\big)^9, \tag{8}$$

where the symmetric matrix $\mathbf{E}$ is equal to

$$\begin{pmatrix} S_3\frac{\partial \mathbf{R}}{\partial s_1}\mathbf{n}\cdot\mathbf{R}\mathbf{t}_1 + \mathcal{Z}_1\cdot\mathbf{R}\mathbf{t}_1 & S_3\frac{\partial \mathbf{R}}{\partial s_1}\mathbf{n}\cdot\mathbf{R}\mathbf{t}_2 + \frac{1}{2}\{\mathcal{Z}_2\cdot\mathbf{R}\mathbf{t}_1 + \mathcal{Z}_1\cdot\mathbf{R}\mathbf{t}_2\} & \frac{1}{2}\frac{\partial \bar{v}}{\partial S_3}\cdot\mathbf{R}\mathbf{t}_1 + \frac{1}{2}\mathcal{Z}_1\cdot\mathbf{R}\mathbf{n} \\ * & S_3\frac{\partial \mathbf{R}}{\partial s_2}\mathbf{n}\cdot\mathbf{R}\mathbf{t}_2 + \mathcal{Z}_2\cdot\mathbf{R}\mathbf{t}_2 & \frac{1}{2}\frac{\partial \bar{v}}{\partial S_3}\cdot\mathbf{R}\mathbf{t}_2 + \frac{1}{2}\mathcal{Z}_2\cdot\mathbf{R}\mathbf{n} \\ * & * & \frac{\partial \bar{v}}{\partial S_3}\cdot\mathbf{R}\mathbf{n} \end{pmatrix}$$

and where $(\mathbf{t}_1 \mid \mathbf{t}_2 \mid \mathbf{n})$ denotes the 3×3 matrix with first column $\mathbf{t}_1$, second column $\mathbf{t}_2$, and third column $\mathbf{n}$, and where $(\mathbf{t}_1 \mid \mathbf{t}_2 \mid \mathbf{n})^{-T} = ((\mathbf{t}_1 \mid \mathbf{t}_2 \mid \mathbf{n})^{-1})^T$.

If the sequence (v_δ) satisfies $\|\operatorname{dist}(\nabla_x v, SO(3))\|_{L^2(\mathcal{Q}_\delta)} \leqslant C\delta^{3/2}$ where C does not depend on δ, the convergences (7) and (8) still hold. The quantities $\mathcal{Z}_\alpha$ measure the defect between a pure bending deformation (satisfying $\frac{\partial \mathcal{V}}{\partial s_\alpha} = \mathbf{R}\mathbf{t}_\alpha$) and the mid-surface deformation $\mathcal{V}_\delta$.

6. Limit of the Green–St Venant's strain tensor for $\|\operatorname{dist}(\nabla_x v, SO(3))\|_{L^2(\mathcal{Q}_\delta)}$ of order $\delta^{5/2}$

For a sequence $(v_\delta) \subset \mathbf{U}_\delta$ of deformations such that $\|\operatorname{dist}(\nabla_x v_\delta, SO(3))\|_{L^2(\mathcal{Q}_\delta)} \leqslant C\delta^{5/2}$ we obtain:

Lemma 6. *There exists a subsequence still indexed by δ such that*

$$\begin{cases} \forall X \in \mathbb{R}^3,\ \frac{1}{\delta}(\mathbf{R}_\delta - \mathbf{I}_3)X \rightharpoonup \mathcal{R}\wedge X \text{ weakly in } (H^1(\omega))^3, \\ \frac{1}{\delta}(\mathcal{V}_\delta - \phi) \to \mathcal{U} \text{ strongly in } (H^1(\omega))^3, \\ \frac{1}{\delta^3}\Pi_\delta \bar{v}_\delta \rightharpoonup \bar{v} \text{ weakly in } (L^2(\omega; H^1(-1,1)))^3, \\ \frac{1}{\delta^2}\big(\frac{\partial \mathcal{V}_\delta}{\partial s_\alpha} - \mathbf{R}_\delta \mathbf{t}_\alpha\big) \rightharpoonup \mathcal{Z}_\alpha \text{ weakly in } (L^2(\omega))^3, \end{cases} \tag{9}$$

and for all $X \in \mathbb{R}^3$,

$$\begin{cases} \frac{1}{\delta}\Pi_\delta(v_\delta - \Phi) \to \mathcal{U} \text{ strongly in } (H^1(\Omega))^3, \\ \frac{1}{\delta}\Pi_\delta(\nabla_x v_\delta - \mathbf{I}_3)X \to \mathcal{R}\wedge X \text{ strongly in } (L^2(\Omega))^9, \end{cases} \tag{10}$$

where $\mathcal{R} \in (H^1(\omega))^3$, $\mathcal{U} \in (H^2(\omega))^3$, $\bar{v} \in (L^2(\omega; H^1(-1,1)))^3$ and $\mathcal{Z}_\alpha \in (L^2(\omega))^3$. Moreover, we have $\mathcal{U} = \mathcal{R} = 0$ on γ_0, and $\frac{\partial \mathcal{U}}{\partial s_\alpha} = \mathcal{R}\wedge\mathbf{t}_\alpha$.

Now we equip $(H^1_{\gamma_0}(\omega))^3 = \{U \in (H^1(\omega))^3 \mid U = 0 \text{ on } \gamma_0\}$ with its standard inner product. We let $e_{\alpha\beta}(U) = \frac{1}{2}\{\frac{\partial U}{\partial s_\alpha} \cdot \mathbf{t}_\beta + \frac{\partial U}{\partial s_\beta} \cdot \mathbf{t}_\alpha\}$ and $D_{In} = \{U \in (H^1_{\gamma_0}(\omega))^3 \mid e_{11}(U) = e_{12}(U) = e_{22}(U) = 0\}$. Let $D_{Ex} = (D_{In})^\perp$ denote the orthogonal complement of D_{In} in $(H^1_{\gamma_0}(\omega))^3$. An element of D_{In} is an *inextensional* displacement, while an element of D_{Ex} is an *extensional* one.

We equip D_{Ex} with the norm $\|U\|_{Ex} = \sqrt{\|e_{11}(U)\|^2_{L^2(\omega)} + \|e_{12}(U)\|^2_{L^2(\omega)} + \|e_{22}(U)\|^2_{L^2(\omega)}}$. Generally D_{Ex} is not a Hilbert space for the above norm. We denote by $\mathcal{D}_{Ex}$ the Hilbert space obtained by completion of D_{Ex} for the norm $\|\cdot\|_{Ex}$, and for $U \in \mathcal{D}_{Ex}$ we still denote by $e_{\alpha\beta}(U)$ the limit obtained by density. But notice that we use here improper notations because the element U has not always derivatives in the distribution sense. For a plate for which $\phi(s_1, s_2) = (s_1, s_2, 0)$, we have $\mathbf{t}_1 = \mathbf{e}_1$, $\mathbf{t}_2 = \mathbf{e}_2$ and $\mathbf{n} = \mathbf{e}_3$, then $D_{Ex} = \mathcal{D}_{Ex} = (H^1_{\gamma_0}(\omega))^2$. Thanks to Korn's inequality in $(H^1_{\gamma_0}(\omega))^2$, the norm $\|\cdot\|_{Ex}$ is equivalent to the H^1-norm. The extensional displacements are the membrane displacements.

Now we write $\mathcal{V}_\delta - \phi = U_{I,\delta} + U_{E,\delta}$ where $U_{I,\delta} \in D_{In}$ and $U_{E,\delta} \in D_{Ex}$. Using estimates (2), we obtain $\|U_{E,\delta}\|_{Ex} \leqslant C\delta^2$ and $\|U_{E,\delta}\|_{(H^1(\omega))^3} \leqslant C\delta$. With these notations, the behavior of the strain tensor is given in the following theorem:

Theorem 7. *There exists a subsequence still indexed by δ such that*

$$\frac{1}{\delta^2} U_{E,\delta} \rightharpoonup U_E \quad \text{weakly in } \mathcal{D}_{Ex}.$$

Let $\mathcal{Z}_{\alpha\beta} = e_{\alpha\beta}(U_E) + \frac{1}{2}\frac{\partial \mathcal{U}}{\partial s_\alpha} \cdot \frac{\partial \mathcal{U}}{\partial s_\beta}$ and $\bar{u} = \bar{v} + \frac{S_3}{2}(\mathcal{Z}_1 \cdot \mathbf{n})\mathbf{t}'_1 + \frac{S_3}{2}(\mathcal{Z}_2 \cdot \mathbf{n})\mathbf{t}'_2$, where $(\mathbf{t}'_1, \mathbf{t}'_2)$ is the contravariant basis of $(\mathbf{t}_1, \mathbf{t}_2)$. For this subsequence, we have

$$\frac{1}{2\delta^2} \Pi_\delta\big((\nabla_x v_\delta)^T \nabla_x v_\delta - \mathbf{I}_3\big) \rightharpoonup \big(\mathbf{t}_1 \mid \mathbf{t}_2 \mid \mathbf{n}\big)^{-T} \mathbf{E} \big(\mathbf{t}_1 \mid \mathbf{t}_2 \mid \mathbf{n}\big)^{-1} \quad \text{weakly in } \big(L^1(\Omega)\big)^9, \tag{11}$$

where the symmetric matrix $\mathbf{E}$ is defined by

$$\mathbf{E} = \begin{pmatrix} S_3[\frac{\partial \mathcal{R}}{\partial s_1} \wedge \mathbf{n}] \cdot \mathbf{t}_1 + \mathcal{Z}_{11} & S_3[\frac{\partial \mathcal{R}}{\partial s_1} \wedge \mathbf{n}] \cdot \mathbf{t}_2 + \mathcal{Z}_{12} & \frac{1}{2}\frac{\partial \bar{u}}{\partial S_3} \cdot \mathbf{t}_1 \\ * & S_3[\frac{\partial \mathcal{R}}{\partial s_2} \wedge \mathbf{n}] \cdot \mathbf{t}_2 + \mathcal{Z}_{22} & \frac{1}{2}\frac{\partial \bar{u}}{\partial S_3} \cdot \mathbf{t}_2 \\ * & * & \frac{\partial \bar{u}}{\partial S_3} \cdot \mathbf{n} \end{pmatrix}.$$

The expression of the limit strain (8) (resp. (11)) simplifies the derivation of the Γ-limit of the total energy for elastic material. The limit energy is a functional of $\mathcal{V}$, $\mathbf{R}$ and $\bar{v}$ (resp. of $\mathcal{U}$, $\mathcal{R}$, U_E and $\bar{u}$), the fields $\mathcal{Z}_\alpha$ being eliminated in the Γ-limit process.

References

[1] D. Blanchard, G. Griso, Decomposition of deformations of thin rods. Application to nonlinear elasticity, Anal. Appl. 7 (1) (2009) 21–71.

[2] P.G. Ciarlet, Mathematical Elasticity, Theory of Shells, vol. III, North-Holland, Amsterdam, 2000.

[3] P.G. Ciarlet, L. Gratie, C. Mardare, A nonlinear Korn inequality on a surface, J. Math. Pures Appl. 85 (2006) 2–16.

[4] P.G. Ciarlet, C. Mardare, An introduction to shell theory, in: P.G. Ciarlet, T.T. Li (Eds.), Differential Geometry: Theory and Applications, World Scientific, Singapore, 2008, pp. 94–184.

[5] G. Friesecke, R.D. James, S. Müller, A theorem on geometric rigidity and the derivation of nonlinear plate theory from the three-dimensional elasticity, Comm. Pure Appl. Math. LV (2002) 1461–1506.

[6] G. Friesecke, R.D. James, S. Müller, A hierarchy of plate models derived from nonlinear elasticity by Γ-convergence, Arch. Rational Mech. Anal. 130 (2006) 183–236.

[7] G. Friesecke, R.D. James, M.G. Mora, S. Müller, Derivation of nonlinear bending theory for shells from three-dimensional nonlinear elasticity by Gamma convergence, C. R. Acad. Sci. Paris, Ser. I 336 (2003) 697–702.

[8] G. Griso, Decomposition of displacements of thin structures, J. Math. Pures Appl. 89 (2008) 199–233.

[9] G. Griso, Asymptotic behavior of structures made of plates, Anal. Appl. 3 (4) (2005) 325–356.

[10] O. Pantz, On the justification of the nonlinear inextensional plate model, Arch. Rational Mech. Anal. 167 (3) (2003) 179–209.