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A new characterisation of idempotent states on finite and compact quantum groups

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Abstract

We show that idempotent states on finite quantum groups correspond to pre-subgroups in the sense of Baaj, Blanchard, and Skandalis. It follows that the lattices formed by the idempotent states on a finite quantum group and by its coidalgebras are isomorphic. We show, furthermore, that these lattices are also isomorphic for compact quantum groups, if one restricts to expected coidalgebras. ***To cite this article: U. Franz, A. Skalski, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Une nouvelle caractérisation des états idempotents sur des groupes quantiques finis ou compacts. Nous donnons une caractérisation des états idempotents sur un groupe quantique fini en termes des pré-sous-groupes introduits par Baaj, Blanchard, et Skandalis, et en déduisons un isomorphisme entre le réseau des états idempotents et le réseau des sous-algèbres coïdéales d'un groupe quantique fini. Cet isomorphisme s'étend aux groupes quantiques compacts, si on le restreind au sous-algèbres coïdéales expectées. ***Pour citer cet article : U. Franz, A. Skalski, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Version française abrégée

Kawada et Itô [8] ont montré que toute mesure idempotente sur un groupe compact est induite par la mesure de Haar d'un de ses sous-groupes compacts, voir aussi [6]. Depuis Pal [11] nous savons que l'analogue de ce théorème pour les groupes quantiques est fausse. Recemment nous avons donné de nouvelles examples d'états idempotents sur des groupes quantiques finis qui ne sont pas induits par des états de Haar de sous-groupes quantiques, voir [4]. Nous en avons également donné une caractérisation en termes de sous-hypergroupes quantiques.

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Dans cette Note nous donnons deux nouvelles caractérisations des états idempotents sur les groupes quantiques finis.

La première ressemble au résultat classique de Kawada et Itô, mais il fallait remplacer les sous-groupes quantiques par les pré-sous-groupes [1]. La deuxième, en terme de sous-algèbres coïdeales, découle ensuite d'un résultat de Baaj, Blanchard et Skandalis. Cette deuxième charactérisation s'étend aussi aux groupes quantiques compacts.

Rappelons qu'un *groupe quantique compact* est une C^*-algèbre unifère A munie d'un $*$-homomorphisme $\Delta : \mathsf{A} \to \mathsf{A} \otimes \mathsf{A}$ dite *coproduit* tel que $(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta$ et les espaces $\mathrm{span}\{(\mathbf{1} \otimes a)\Delta(b); a, b \in \mathsf{A}\}$ et $\mathrm{span}\{(a \otimes \mathbf{1})\Delta(b); a, b \in \mathsf{A}\}$ sont denses dans $\mathsf{A} \otimes \mathsf{A}$, cf. [14,15]. Si A est à dimension finie, on parle de *groupe quantique fini*. Le coproduit permet de définir un produit de convolution $\psi_1 \star \psi_2 = (\psi_1 \otimes \psi_2) \circ \Delta$ pour $\psi_1, \psi_2 : \mathsf{A} \to \mathbb{C}$. Un état $\phi : \mathsf{A} \to \mathbb{C}$ est dit *idempotent*, si $\phi \star \phi = \phi$. Nous l'appelons *état idempotent de type Haar*, s'il peut s'écrire comme $\phi = h_{\mathsf{B}} \circ \pi$, où $(\mathsf{B}, \Delta_{\mathsf{B}})$ est un sous-groupe quantique de (A, Δ), avec morphisme $\pi : \mathsf{A} \to \mathsf{B}$ et état de Haar $h_{\mathsf{B}} : \mathsf{B} \to \mathbb{C}$. L'example de Pal [11] montre qu'il existe des états idempotents sur des groupes quantiques qui ne peuvent pas s'écrire sous cette forme.

Soit $H = L^2(\mathsf{A}, h)$ l'espace hilbertien sous-jacent de la représentation GNS de A par rapport à l'état de Haar h. Rappelons qu'un pré-sous-groupe de A est un vecteur $f \in H$ de norme $||f|| = 1$ tel que $\varepsilon(f) > 0$ et $V(f \otimes f) = f \otimes f$ (où ε est la counité et $V : H \otimes H \to H \otimes H$ l'opérateur unitaire défini par $V(a \otimes b) = \Delta(a)(\mathbf{1} \otimes b)$ pour $a, b \in \mathsf{A} \subseteq H$).

Posons $\omega_{u,v} : \mathsf{A} \to \mathbb{C}$, $\omega_{u,v}(a) = \langle u, av \rangle = h(u^* av)$ pour $u, v \in H$.

Théorème 0.1. *Soit* (A, Δ) *un groupe quantique fini. Alors* $f \mapsto \omega_{f,f}$ *définit une bijection entre le réseau des pré-sous-groupes de* (A, Δ) *et le réseau des états idempotents sur* (A, Δ).

Grace à [1, Proposition 4.3], on peut en déduire la caractérisation suivante :

Corollaire 0.2. *Soit* (A, Δ) *un groupe quantique fini. Alors le réseau des sous-algèbres coïdéales à droite de* (A, Δ) *et le réseau des états idempotents sur* (A, Δ) *sont isomorphes.*

Cette deuxième caractérisation se généralise aux groupes quantiques compacts, si on impose l'existence d'une espérance conditionnelle.

Théorème 0.3. *Soit* (A, Δ) *un groupe quantique compact comoyennable. Alors le réseau des sous-algèbres expectées coïdéales à droite de* (A, Δ) *et le réseau des états idempotents sur* (A, Δ) *sont isomorphes.*

1. Introduction

The idempotent measures on a locally compact group are exactly the Haar measures of its compact subgroups, cf. [6,8]. In 1996, Pal [11] has shown that the analogous statement for quantum groups is false. In [4], we have given more examples of idempotent states on quantum groups that do not come from compact subgroups. We also provided characterisations of idempotent states on finite quantum groups in terms of group-like projections [9] and quantum subhypergroups. Subsequently with Tomatsu we extended some of these results to compact quantum groups, and determined all idempotent states on the compact quantum groups $U_q(2)$, $SU_q(2)$, and $SO_q(3)$, cf. [5].

In this Note we give a new characterisation of idempotent states on finite quantum groups in terms of the pre-subgroups introduced in [1]. That pre-subgroups give rise to idempotent states was not emphasized in [1], but can easily be seen from [1, Proposition 3.5(a)]. Here we prove that, conversely, every idempotent state comes from a pre-subgroup, cf. Theorem 3.2. As a consequence, we get a one-to-one correspondence between the idempotent states on a finite quantum group (A, Δ) and the coidalgebras in (A, Δ), cf. Corollary 3.4. The isomorphisms providing this bijection have natural explicit descriptions, cf. Remark 1 after Corollary 3.4. The idempotent states coming from compact quantum subgroups are exactly those corresponding to subgroups in the sense of Baaj, Blanchard, and Skandalis, and to coidalgebras of quotient type, see Proposition 3.6.

The one-to-one correspondence between idempotent states and coidalgebras extends to compact quantum groups, if one restricts to expected coidalgebras, cf. Theorem 4.1.

2. Preliminaries

Recall that a *compact quantum group* is a pair (A, Δ) of a unital C^*-algebra A and a unital $*$-homomorphism $\Delta : \mathsf{A} \to \mathsf{A} \otimes \mathsf{A}$ such that $(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta$ holds, and the subspaces

$$\mathrm{span}\{(\mathbf{1} \otimes a)\Delta(b); a, b \in \mathsf{A}\} \quad \text{and} \quad \mathrm{span}\{(a \otimes \mathbf{1})\Delta(b); a, b \in \mathsf{A}\}$$

are dense in $\mathsf{A} \otimes \mathsf{A}$, cf. [14,15] (here $\otimes$ denotes the minimal tensor product of C^*-algebras reducing to the algebraic tensor product in the finite-dimensional situation). If A is finite-dimensional, then (A, Δ) is called a *finite quantum group*. Woronowicz showed that there exists a unique state $h : \mathsf{A} \to \mathbb{C}$ such that

$$(\mathrm{id}_{\mathsf{A}} \otimes h) \circ \Delta(a) = h(a)\mathbf{1} = (h \otimes \mathrm{id}_{\mathsf{A}}) \circ \Delta(a) \quad \text{for all } a \in \mathsf{A},$$

called the Haar state of (A, Δ). If (A, Δ) is a finite quantum group, then h is a faithful trace. A finite quantum group has a unique *counit*, i.e. a character $\varepsilon : \mathsf{A} \to \mathbb{C}$ such that $(\varepsilon \otimes \mathrm{id}_{\mathsf{A}}) \circ \Delta = \mathrm{id}_{\mathsf{A}} = (\mathrm{id}_{\mathsf{A}} \otimes \varepsilon) \circ \Delta$, and a unique *Haar element*, i.e. a projection $\eta \in \mathsf{A}$ such that $\eta a = a\eta = \varepsilon(a)\eta$ for all $a \in \mathsf{A}$. For more information on finite-dimensional $*$-Hopf algebras and their Haar states, see [13].

Define $V : \mathsf{A} \otimes \mathsf{A} \to \mathsf{A} \otimes \mathsf{A}$ by $V(a \otimes b) = \Delta(a)(\mathbf{1} \otimes b)$. Then V extends to a unitary operator $V : H \otimes H \to H \otimes H$ ($H = L^2(\mathsf{A}, h)$ denotes the GNS Hilbert space of the Haar state), which satisfies $V_{12}V_{13}V_{23} = V_{23}V_{12}$, on $H \otimes H \otimes H$, where we used the leg notation $V_{12} = V \otimes \mathrm{id}$, etc. The operator V is called the *multiplicative unitary* of (A, Δ), see also [2].

The notion of a quantum subgroup was introduced by Kac [7] in the setting of finite ring groups and by Podleś [12] for matrix pseudo-groups.

Definition 2.1. Let $(\mathsf{A}, \Delta_{\mathsf{A}})$ and $(\mathsf{B}, \Delta_{\mathsf{B}})$ be two compact quantum groups. Then $(\mathsf{B}, \Delta_{\mathsf{B}})$ is called a *quantum subgroup* of $(\mathsf{A}, \Delta_{\mathsf{A}})$, if there is exists a surjective $*$-algebra homomorphism $\pi : \mathsf{A} \to \mathsf{B}$ such that $\Delta_{\mathsf{B}} \circ \pi = (\pi \otimes \pi) \circ \Delta_{\mathsf{A}}$.

This definition is motivated by the properties of the restriction map $C(G) \ni f \mapsto f|_H \in C(H)$ induced by a subgroup $H \subseteq G$. If $\mathsf{A} = C(G)$ is a commutative compact quantum group, then Definition 2.1 is equivalent to the usual notion of a closed subgroup.

Definition 2.2. (See [1, Definition 3.4].) Let $(\mathsf{A}, \Delta_{\mathsf{A}})$ be a finite quantum group with multiplicative unitary $V : H \otimes H \to H \otimes H$. Then a *pre-subgroup* of $(\mathsf{A}, \Delta_{\mathsf{A}})$ is a unit vector $f \in H$ such that $\varepsilon(f) > 0$, and $V(f \otimes f) = f \otimes f$.

Denote by $\mathbf{1}_h \in H$ the cyclic vector that implements the Haar state. For a finite quantum group, $\mathsf{A} \ni a \mapsto a\mathbf{1}_h \in H$ is an isomorphism and $\varepsilon(f)$ is to be understood via this identification.

We will frequently use this identification and omit $\mathbf{1}_h$ in the rest of the paper.

A pre-subgroup f is called a *subgroup*, if it belongs to the center of A. In that case f gives rise to a quantum subgroup in the sense of Definition 2.1, cf. Lemma 3.5.

A non-zero element $p \in \mathsf{A}$ in a compact quantum group (A, Δ) is called a *group-like projection* [9, Definition 1.1], if it is a projection, i.e. $p^2 = p = p^*$, and satisfies $\Delta(p)(\mathbf{1} \otimes p) = p \otimes p$. We shall see that for finite quantum groups pre-subgroups and group-like idempotents are essentially the same objects, i.e. that after a rescaling pre-subgroups are group-like projections in A, cf. Corollary 3.3.

For commutative finite quantum groups of the form $\mathsf{A} = C(G)$, pre-subgroups are multiples of indicator functions of subgroups, cf. [9, Proposition 1.4], but for noncommutative finite quantum groups this notion is more general than Definition 2.1.

Baaj, Blanchard, and Skandalis defined an order of pre-subgroups by $g \prec f$ if and only if $V(f \otimes g) = f \otimes g$.

3. Characterisations of idempotents states on finite quantum groups

The coproduct $\Delta : \mathsf{A} \to \mathsf{A} \otimes \mathsf{A}$ leads to an associative product $\psi_1 \star \psi_2 = (\psi_1 \otimes \psi_2) \circ \Delta$ called the *convolution product*, for linear functionals $\psi_1, \psi_2 : \mathsf{A} \to \mathbb{C}$. A state $\phi : \mathsf{A} \to \mathbb{C}$ is *idempotent*, if $\phi \star \phi = \phi$. Examples are given by $\phi = h_{\mathsf{B}} \circ \pi$, if $(\mathsf{B}, \Delta_{\mathsf{B}})$ is a quantum subgroup of $(\mathsf{A}, \Delta_{\mathsf{A}})$ with morphism $\pi : \mathsf{A} \to \mathsf{B}$ and Haar state $h_{\mathsf{B}} : \mathsf{B} \to \mathbb{C}$. We will call an idempotent state ϕ on a compact quantum group (A, Δ) a *Haar idempotent state*, if it is of this form.

The natural order for projections can be used to equip the set of idempotent states on a compact quantum group with a partial order, i.e. $\phi_1 \prec \phi_2$ if and only if $\phi_1 \star \phi_2 = \phi_2$, cf. [4, Section 5].

Before we can state and prove the main theorem, we need the following lemma, which is a slight variation of [10, Lemma 4.3]:

Lemma 3.1. *Let* (A, Δ) *be a compact quantum group with two states* f *and* g *such that* $g \star f = f \star g = f$. *Denote by* g_b *the functional defined by* $g_b(a) = g(ab)$ *for* $a, b \in \mathsf{A}$. *Then we have* $f \star g_b = g(b) f$ *for all* $b \in \mathsf{A}$.

For $u, v \in L^2(\mathsf{A}, h)$, denote by $\omega_{u,v} : \mathsf{A} \to \mathbb{C}$ the linear functional $\mathsf{A} \ni a \mapsto \omega_{u,v}(a) = \langle u, av\rangle = h(u^* av)$.

We have the following characterisation of idempotent states in terms of pre-subgroups:

Theorem 3.2. *Let* (A, Δ) *be a finite quantum group. Then the map* $f \mapsto \omega_{f,f}$ *defines an order-preserving bijection between the pre-subgroups of* (A, Δ) *and the idempotent states on* (A, Δ).

Proof. Let $\omega_{f,f}$ be the state associated to a pre-subgroup $f \in \mathsf{A}$. We have

$$(\omega_{f,f} \star \omega_{f,f})(a) = \langle f \otimes f, \Delta(a)(f \otimes f)\rangle = \langle f \otimes f, V(a \otimes \mathbf{1})V^*(f \otimes f)\rangle$$
$$= \langle f \otimes f, (a \otimes \mathbf{1})(f \otimes f)\rangle = \omega_{f,f}(a),$$

for all $a \in \mathsf{A}$, i.e. $\omega_{f,f}$ is an idempotent state. This also follows from [1, Proposition 3.5(a)].

Conversely, let $\phi : \mathsf{A} \to \mathbb{C}$ be an idempotent state. Since the Haar state is tracial, there exists a unique positive element $\rho_\phi \in \mathsf{A}$ such that $\phi(a) = \langle \rho_\phi, a\rangle$ for all $a \in \mathsf{A}$. Set $f_\phi = \sqrt{\rho_\phi}$. Then have $\phi(a) = \langle f_\phi, a f_\phi\rangle$ for all $a \in \mathsf{A}$, and $f_\phi = \sqrt{\rho_\phi}$ is the unique positive element with this property.

By Lemma 3.1, we have $\phi \star \phi_b = \phi(b)\phi$, i.e.

$$\langle \rho_\phi \otimes \rho_\phi, a \otimes b\rangle = \phi(a)\phi(b) = (\phi \star \phi_b)(a) = \langle \rho_\phi \otimes \rho_\phi, \Delta(a)(\mathbf{1} \otimes b)\rangle$$
$$= \langle \rho_\phi \otimes \rho_\phi, V(a \otimes \mathbf{1})V^*(\mathbf{1} \otimes b)\rangle = \langle V^*(\rho_\phi \otimes \rho_\phi), a \otimes b\rangle \tag{1}$$

for all $a, b \in \mathsf{A}$, since $V(\mathbf{1} \otimes b) = \Delta(\mathbf{1})(\mathbf{1} \otimes b) = \mathbf{1} \otimes b$. Therefore we have $V(\rho_\phi \otimes \rho_\phi) = \rho_\phi \otimes \rho_\phi$. Recalling the definition of V and the identification between H and A, this means $\Delta(\rho_\phi)(\mathbf{1} \otimes \rho_\phi) = \rho_\phi \otimes \rho_\phi$. Applying ε to the left-hand side, we get $\rho_\phi^2 = \varepsilon(\rho_\phi)\rho_\phi$. Therefore $\varepsilon(\rho_\phi) > 0$ and $f_\phi = \sqrt{\rho_\phi} = \frac{\rho_\phi}{\sqrt{\varepsilon(\rho_\phi)}}$. Clearly, f_ϕ is a unit vector, $\varepsilon(f_\phi) = \sqrt{\varepsilon(\rho_\phi)} > 0$, and $V(f_\phi \otimes f_\phi) = f_\phi \otimes f_\phi$, i.e. f_ϕ is a pre-subgroup.

Let g be another pre-subgroup with $\phi = \omega_{g,g}$. If we can show $g \geqslant 0$, then this implies $g = f_\phi$. Applying ε to $\Delta(g)(\mathbf{1} \otimes g) = g \otimes g$, we get $g^2 = \varepsilon(g)g$. Applying ϕ to the Haar element η, we see $\varepsilon(g) = \varepsilon(f_\phi)$. Furthermore, $\omega_{g,g} = \omega_{f_\phi, f_\phi}$ is equivalent to $gg^* = f_\phi f_\phi^*$. Therefore we get $\|g\| = \|f_\phi\|$, and $g/\varepsilon(g)$ is an idempotent with norm one. Therefore g is an orthogonal projection, in particular positive, and we see that $f \mapsto w_{f,f}$ defines indeed a bijection.

Let now f, g be two pre-subgroups such that $g \prec f$, i.e. $V(f \otimes g) = f \otimes g$. Then

$$(\omega_{f,f} \star \omega_{g,g})(a) = \langle f \otimes g, \Delta(a)(f \otimes g)\rangle = \langle f \otimes g, V(a \otimes \mathbf{1})V^*(f \otimes g)\rangle$$
$$= \langle f \otimes g, (a \otimes \mathbf{1})(f \otimes g)\rangle = \omega_{f,f}(a)$$

for all $a \in \mathsf{A}$, i.e. $\omega_{g,g} \prec \omega_{f,f}$. Conversely, if $\omega_{g,g} \prec \omega_{f,f}$, then $\omega_{g,g} \star \omega_{f,f} = \omega_{f,f}$ by [4, Lemma 5.2], and $\omega_{f,f} \star (\omega_{g,g})_b = \omega_{g,g}(b)\omega_{f,f}$ for all $b \in \mathsf{A}$ by Lemma 3.1. A calculation similar to (1) yields $g \prec f$. □

Corollary 3.3. *Let* (A, Δ) *be a finite quantum group. The map* $f \mapsto \frac{f}{\varepsilon(f)}$ *defines a bijection between the pre-subgroups and the group-like projections of* (A, Δ).

A *right coidalgebra* C in a compact quantum group is a unital $*$-subalgebra $\mathsf{C} \subseteq \mathsf{A}$ such that $\Delta(\mathsf{C}) \subseteq \mathsf{A} \otimes \mathsf{C}$. Baaj, Blanchard, and Skandalis have shown that the lattice of pre-subgroups of a finite quantum group is isomorphic to its lattice of right coidalgebras, cf. [1, Proposition 4.3].

Corollary 3.4. *Let* (A, Δ) *be a finite quantum group. Then the lattice of idempotent states on* (A, Δ) *and the lattice of right coidalgebras in* (A, Δ) *are isomorphic.*

Remark 1. We can also give an explicit description of this bijection. Let $\phi : \mathsf{A} \to \mathbb{C}$ be an idempotent state. Then one can show that $T_\phi : \mathsf{A} \to \mathsf{A}$, $T_\phi = (\mathrm{id}_\mathsf{A} \otimes \phi) \circ \Delta$ defines a conditional expectation, i.e. a projection $E : \mathsf{A} \to \mathsf{C}$ onto a unital $*$-subalgebra $\mathsf{C} \subseteq \mathsf{A}$ such that $\|E\| = 1$, $E(\mathbf{1}) = \mathbf{1}$, and $h \circ E = h$. Furthermore, since T_ϕ is right-invariant, $T_\phi(\mathsf{A})$ is a coidalgebra. Conversely, to recover an idempotent state ϕ from a right coidalgebra $\mathsf{C} \subseteq \mathsf{A}$, set $\phi = \varepsilon \circ E_\mathsf{C}$, where E_C denotes the unique h-preserving conditional expectation onto C. See also Theorem 4.1.

Lemma 3.5. *Let* (A, Δ) *be a finite quantum group, f a subgroup of* (A, Δ)*, i.e. a pre-subgroup that belongs to the center of* A*, and put* $\tilde{f} = \frac{f}{\varepsilon(f)}$*. Then* (A_f, Δ_f) *is a quantum subgroup of* (A, Δ)*, with* $\mathsf{A}_f = \mathsf{A}f = \{af; a \in \mathsf{A}\}$*, and* $\Delta_f : \mathsf{A}_f \to \mathsf{A}_f \otimes \mathsf{A}_f$ *and* $\pi_f : \mathsf{A} \to \mathsf{A}_f$ *given by* $\Delta_f(a) = \Delta(a)(\tilde{f} \otimes \tilde{f})$ *and* $\pi(a) = a\tilde{f}$ *for* $a \in \mathsf{A}$.

Proof. This follows from Corollary 3.3 and [9, Proposition 2.1]. □

For any quantum subgroup $(\mathsf{B}, \Delta_\mathsf{B})$ of (A, Δ), $\mathsf{A}//\mathsf{B} = \{a \in \mathsf{A}; ((\pi \otimes \mathrm{id}) \circ \Delta_\mathsf{A})(a) = \mathbf{1}_\mathsf{B} \otimes a\}$ defines a right coidalgebra. A right coidalgebra is said to be of *quotient type*, if it is of this form.

Using the previous Lemma, one can check that under the one-to-one correspondences given in Theorem 3.2 and Corollary 3.4, Haar idempotent states correspond to subgroups and coidalgebras of quotient type.

Proposition 3.6. *Let ϕ be an idempotent state. Then the following are equivalent*:

(i) *The state ϕ is a Haar idempotent state*;
(ii) *The pre-subgroup f_ϕ is a subgroup*;
(iii) *The coidalgebra C_ϕ is of quotient type.*

4. Extension to compact quantum groups

For a compact quantum group (A, Δ), in general the Haar state h is no longer a trace, and for a closed unital $*$-subalgebra $\mathsf{B} \subseteq \mathsf{A}$ there might exist no h-preserving conditional expectation $E_\mathsf{B} : \mathsf{A} \to \mathsf{B}$. It turns out that the existence of such a conditional expectation is the condition we have to add to extend the bijection between idempotent states and right coidalgebras. Recall that a compact quantum group is called coamenable if its reduced version is isomorphic to the universal one (equivalently, the Haar state h is faithful and A admits a character, cf. [3, Corollary 2.9]). In particular every coamenable compact quantum group admits a counit.

Theorem 4.1. *Let* (A, Δ) *be a coamenable compact quantum group. Then there exists an order-preserving bijection between the expected right coidalgebras in* (A, Δ) *and the idempotent states on* (A, Δ).

Sketch of proof. Given an idempotent state $\phi \in \mathsf{A}^*$ we define a completely positive idempotent projection $E_\phi = (\mathrm{id}_\mathsf{A} \otimes \phi) \circ \Delta$. An application of Lemma 3.1 shows that $E_\phi(E_\phi(a)E_\phi(b)) = E_\phi(a)E_\phi(b)$ for all $a, b \in \mathcal{A}$, where $\mathcal{A}$ is the $*$-Hopf algebra spanned by coefficients of the unitary corepresentations of A. Density of $\mathcal{A}$ in A and the continuity argument implies that $E_\phi(\mathsf{A})$ is an algebra; the right invariance of E_ϕ expressed by the equality $\Delta \circ E_\phi = (\mathrm{id}_\mathsf{A} \otimes E_\phi) \circ \Delta$ implies that $E_\phi(\mathsf{A})$ is a right coidalgebra.

Conversely, if C is an expected right coidalgebra, let E_C denote the corresponding conditional expectation. We can show that if $\mathsf{C}' = \{b \in \mathsf{A}:\ E_\mathsf{C}(b) = 0\}$, then for all $\omega \in \mathsf{A}^*$, $b \in \mathsf{C}'$, $(\omega \otimes \mathrm{id}_\mathsf{A})(\Delta(b)) \in \mathsf{C}'$. This implies that E_C is right invariant and thus $E_\mathsf{C} = (\mathrm{id}_\mathsf{A} \otimes \phi) \circ \Delta$ for the idempotent state $\phi := \varepsilon \circ E_\mathsf{C}$. □

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