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Partial Differential Equations

Remarks on bounded solutions of steady Hamilton–Jacobi equations

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Abstract

We study here the equation $H(Du) = H(0),\ x \in \mathbb{R}^N$. More precisely we investigate under which hypotheses the constant functions are the only bounded solutions. In arbitrary space dimension we prove that this happens when convexity and coercivity occur. In one space dimension we show that the above property holds true for Hamiltonians in a larger class. These results apply when studying the long time behaviour of solutions for time-dependent Hamilton–Jacobi equations. ***To cite this article: M. Bostan, G. Namah, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Quelques remarques sur les solutions bornées des équations stationnaires d'Hamilton–Jacobi. Dans cette Note on s'intéresse à l'équation $H(Du) = H(0),\ x \in \mathbb{R}^N$ et plus précisément à la question suivante : dans quels cas les fonctions constantes sont-elles les seules solutions bornées de cette équation ? On démontre que tel est le cas sous des hypothèses de convexité et coercivité en dimension N quelconque. La preuve fait appel à la formule de Hopf–Lax. En une dimension d'espace on propose un résultat pour des hamiltoniens seulement *faiblement coercifs* moyennant une condition supplémentaire. Dans la dernière partie on utilise ces résultats pour identifier les limites asymptotiques en temps long des solutions des problèmes de Cauchy. ***Pour citer cet article : M. Bostan, G. Namah, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

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Soit $H : \mathbb{R}^N \to \mathbb{R}$ une fonction vérifiant

$$H \text{ convexe}, \tag{1}$$

$$\lim_{|p| \to +\infty} \frac{H(p)}{|p|} = +\infty. \tag{2}$$

On montre que les seules solutions (au sens de viscosité) bornées de l'équation stationnaire d'Hamilton–Jacobi $H(Du) = H(0),\ x \in \mathbb{R}^N$ sont données par les fonctions constantes. Sans perte de généralité on peut supposer que

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$H(0)=0$. La preuve utilise la formule de représentation de Hopf–Lax [4] p. 560. Il est bien connu que si l'hamiltonien H vérifie (1), (2), alors toute solution au sens de viscosité de $H(Du)=0,\ x\in\mathbb{R}^N$ vérifie

$$u(x)=\inf_{y\in\mathbb{R}^N}\left\{u(y)+tL\left(\frac{x-y}{t}\right)\right\} \tag{3}$$

où $L:\mathbb{R}^N\to\mathbb{R}$ est la fonction conjuguée à H par dualité convexe

$$L(q)=\sup_{p\in\mathbb{R}^N}\big\{q\cdot p-H(p)\big\},\quad q\in\mathbb{R}^N. \tag{4}$$

En une dimension d'espace il est possible d'étudier une classe plus large d'hamiltoniens, pas nécessairement convexes. On considère $H:\mathbb{R}\to\mathbb{R}$ seulement faiblement coercif i.e.,

$$\lim_{|p|\to+\infty}H(p)=+\infty \tag{5}$$

et vérifiant

$$0\notin\overline{H^{-1}\big(H(0)\big)\setminus\{0\}}. \tag{6}$$

On voit facilement que si la fonction H reste constante dans un voisinage de 0 alors on peut toujours construire une solution bornée, non constante de $H(u_x)=H(0),\ x\in\mathbb{R}$ (voir l'exemple (14)), ce qui justifie l'introduction de l'hypothèse (6). En utilisant les notions de sous/sur-différentiel (la formule de Hopf–Lax n'étant plus valide, car H n'est plus supposé convexe) on montre que les seules solutions (au sens de viscosité) bornées de

$$H(u_x)=H(0),\quad x\in\mathbb{R} \tag{7}$$

sont les fonctions constantes. Ces résultats s'appliquent lorsqu'on souhaite étudier le comportement en temps long d'un problème d'évolution avec condition initiale

$$\begin{cases}\partial_t u+H(\partial_x u)=0, & (x,t)\in\mathbb{R}\times\,]0,+\infty[,\\ u(x,0)=u_0(x), & x\in\mathbb{R}.\end{cases} \tag{8}$$

1. Bounded stationary solutions

The subject matter of this Note concerns the stationary equation

$$H(Du)=H(0),\quad x\in\mathbb{R}^N \tag{9}$$

where $H:\mathbb{R}^N\to\mathbb{R}$ is a continuous function.

Proposition 1.1. *Let $H=H(p):\mathbb{R}^N\to\mathbb{R}$ satisfying* (1), (2) *such that its conjugate function is C^1 in a neighbourhood of its minimum points. Then the constants are the only bounded solutions of* (9) (*in viscosity sense*).

We appeal here to Hopf–Lax representation formula [4] p. 560. Before detailing the proof let us recall the following standard results concerning the conjugate (by convex duality) function L associated to the Hamiltonian H. We have (see [4] p. 122):

Proposition 1.2 *(Convex duality of Hamiltonian and Lagrangian). Assume that H satisfies* (1), (2)*. Then the mapping L is convex and satisfies $\lim_{|q|\to+\infty}L(q)/|q|=+\infty$. Furthermore, the conjugate function associated do L coincides with H.*

Notice that (2) guarantees that the supremum in (4) is attained for any $q\in\mathbb{R}^N$ and that the mapping L is locally Lipschitz. We check easily that $L(q)\geqslant -H(0),\ q\in\mathbb{R}^N$. If H is strictly convex, the dual function L enjoys other interesting properties, see for e.g. [3]. Indeed, for any $q\in\mathbb{R}^N$, there is a unique $p\in\mathbb{R}^N$ such that $\partial H(p)\ni q$ and we have for any element in $\partial H(p)$

$$L\big(\partial H(p)\big)=\partial H(p)\cdot p-H(p),\quad p\in\mathbb{R}^N. \tag{10}$$

We also mention that the strict convexity of H ensures that L is C^1 function and $DL(\partial H(p)) = p,\ p \in \mathbb{R}^N$.

Proof of Proposition 1.1. Without loss of generality we can assume that $H(0) = 0$. First notice that by the coercivity condition (2) any bounded solution is in fact Lipschitz continuous so that it is a.e. differentiable. Therefore it suffices to show that $Du = 0$ a.e. to conclude that u is a constant function. For this sake we are going to use the representation Hopf–Lax formula (3) for convex coercive Hamiltonians. Let x_0 be a differentiability point of u. We want to show that $Du(x_0) = 0$. Let $y_0^t = y_0^t(x_0, t)$ be a minimum point in (3) i.e.,

$$u(x_0) = u\big(y_0^t\big) + tL\big(z_0^t\big), \quad z_0^t = \frac{x_0 - y_0^t}{t}. \tag{11}$$

Since u is bounded we have

$$L\big(z_0^t\big) \leqslant \frac{2\|u\|_{L^\infty}}{t}, \quad t > 0. \tag{12}$$

The boundedness of L together with its coercivity lead then to the boundedness of $(z_0^t)_{t>0}$. Thus there is a sequence $(t_k)_k$ diverging towards $+\infty$ such that the sequence $(z_0^{t_k})_k$ converges to some limit z_∞. From (12) we deduce that $L(z_\infty) \leqslant 0$. As $L(q) \geqslant -H(0) = 0$ for any $q \in \mathbb{R}^N$, we conclude that $L(z_\infty) = 0$ saying that z_∞ is a minimum point for L, $DL(z_\infty) = 0$ and therefore

$$\lim_{k\to+\infty} DL\big(z_0^{t_k}\big) = DL(z_\infty) = 0. \tag{13}$$

The idea is to differentiate (11) with respect to x_0 for any fixed k and then to let $k \to +\infty$. By the Hopf–Lax formula (3) we have for any $x \in \mathbb{R}^N$

$$u(x) - u\big(y_0^{t_k}\big) - t_k L\left(\frac{x - y_0^{t_k}}{t_k}\right) \leqslant 0 = u(x_0) - u\big(y_0^{t_k}\big) - t_k L\left(\frac{x_0 - y_0^{t_k}}{t_k}\right)$$

saying that the function $x \to u(x) - t_k L(\frac{x-y_0^{t_k}}{t_k})$ has a maximum in x_0. Since x_0 is a differentiability point of u and L is C^1 around its minimum point $z_\infty = \lim_{k\to+\infty} z_0^{t_k}$ we deduce that

$$Du(x_0) = DL\big(z_0^{t_k}\big) \to DL(z_\infty) = 0, \quad \text{as } k \to +\infty. \qquad \Box$$

We now propose a result in one dimension concerning the possible bounded solutions of (7) when $H : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the coercivity condition (5). For this sake consider the set

$$I_H = \big\{p \in \mathbb{R}:\ H(p) = H(0)\big\}.$$

Notice that I_H is nonempty ($0 \in I_H$) and closed. We then have

Proposition 1.3. *Let $H = H(p)$ satisfy* (5) *such that $0 \notin \overline{I_H - \{0\}}$. Then the constants are the only bounded solutions of* (7).

Let us point out that strictly convex Hamiltonians satisfy $0 \notin \overline{I_H - \{0\}}$. Notice also that just convex is not sufficient as is shown by the following example:

$$H(p) = \begin{cases} -p - 1, & p \in]-\infty, -1[, \\ 0, & p \in [-1, 1], \\ p - 1, & p \in]1, \infty[, \end{cases} \tag{14}$$

where possible solutions are (up to additive constants) any function v such that $|v'(x)| \leqslant 1,\ x \in \mathbb{R}$. Here we have $\overline{I_H - \{0\}} = [-1, 1] \ni 0$ and as we see solutions other than constants exist (for e.g. $u(x) = \sin x,\ x \in \mathbb{R}$). The proof of the above proposition relies on the notions of subdifferential and superdifferential for continuous functions $v \in C(\mathbb{R})$

$$D^-v(x)=\left\{p\in\mathbb{R}\text{: }\liminf_{y\to x}\frac{v(y)-v(x)-p(y-x)}{|y-x|}\geqslant 0\right\},$$
$$D^+v(x)=\left\{p\in\mathbb{R}\text{: }\limsup_{y\to x}\frac{v(y)-v(x)-p(y-x)}{|y-x|}\leqslant 0\right\}.$$

We use the following easy lemma:

Lemma 1.4. *Let $v\in C(\mathbb{R})$ be a continuous function and $x_1<x_2$ two differentiability points for v such that $v'(x_1)\neq v'(x_2)$.*

(i) *If $v'(x_1)<k<v'(x_2)$ there is $x_3\in\,]x_1,x_2[$ such that $D^-v(x_3)\ni k$.*
(ii) *If $v'(x_1)>k>v'(x_2)$ there is $x_4\in\,]x_1,x_2[$ such that $D^+v(x_4)\ni k$.*

Proof. Without loss of generality we can assume $k=0$ (replace the function $v(x)$ by $v(x)-kx,\ x\in\mathbb{R}$).

(i) We assume that $v'(x_1)<0<v'(x_2)$. Consider $x_3\in[x_1,x_2]$ such that $v(x_3)=\min\{v(x)\text{: }x\in[x_1,x_2]\}$. Obviously we have $x_3\neq x_1,\ x_3\neq x_2$ and therefore

$$\liminf_{y\to x_3}\frac{v(y)-v(x_3)}{|y-x_3|}\geqslant 0,$$

saying that $0\in D^-v(x_3)$.

(ii) In the case $v'(x_1)>0>v'(x_2)$ take $x_4\in[x_1,x_2]$ such that $v(x_4)=\max\{v(x)\text{: }x\in[x_1,x_2]\}$. We easily check that $0\in D^+v(x_4)$. □

Now we are ready to prove Proposition 1.3.

Proof of Proposition 1.3. Let u be a bounded solution of (7). Since the Hamiltonian satisfies the coercivity condition, u is a Lipschitz function and therefore it is differentiable a.e. on $\mathbb{R}$. We show that the first derivative of u has constant sign. More precisely we prove that if there is $x_1\in\mathbb{R}$ such that $u'(x_1)\neq 0$ then $u'(x)u'(x_1)>0$, for a.a. $x\in\mathbb{R}$. To fix the ideas assume that there is $x_1\in\mathbb{R}$ such that $u'(x_1)<0$, the other case following in similar way. Suppose that there is x_2 such that $u'(x_2)>0$ and let us search for a contradiction.

Case 1. Consider that $x_1<x_2$. By Lemma 1.4 we know that for any $p\in\,]u'(x_1),u'(x_2)[$ there is $x_p\in\,]x_1,x_2[$ such that $D^-u(x_p)\ni p$. Since u is in particular a supersolution of (7) we deduce that

$$H(p)\geqslant H(0),\quad \forall p\in\,\big]u'(x_1),u'(x_2)\big[. \tag{15}$$

We use now the hypothesis $0\notin\overline{I_H-\{0\}}$ which is equivalent to

$$\exists\varepsilon>0\text{:}\quad H(p)\neq H(0),\quad \forall p\in\,]-\varepsilon,0[\,\cup\,]0,\varepsilon[. \tag{16}$$

Obviously since $u'(x_1)<0, u'(x_2)>0$, $H(u'(x_1))=H(u'(x_2))=H(0)$ we obtain $u'(x_1)\leqslant-\varepsilon<\varepsilon\leqslant u'(x_2)$. It remains to notice that $u'(x_3)\geqslant\varepsilon$ for a.a. $x_3>x_2$. Indeed if there is $x_3>x_2$ such that $u'(x_3)<\varepsilon\leqslant u'(x_2)$ by Lemma 1.4 we know that for any $p\in\,]u'(x_3),u'(x_2)[$ there is $y_p\in\,]x_2,x_3[$ such that $D^+u(y_p)\ni p$. Since u is in particular a subsolution of (7) we deduce that

$$H(p)\leqslant H(0),\quad p\in\,\big]u'(x_3),u'(x_2)\big[. \tag{17}$$

Combining (15), (17) we obtain $H(p)=H(0)$ for any

$$p\in\,\big]\max\{u'(x_1),u'(x_3)\},u'(x_2)\big[,$$

which is not possible in view of (16) and of the inequalities $\max\{u'(x_1),u'(x_3)\}<\varepsilon$, $u'(x_2)\geqslant\varepsilon$. Thus the inequality $u'(x_3)\geqslant\varepsilon$ holds for a.a. $x_3>x_2$ leading to a contradiction since in this case $u(x_3)$ becomes unbounded when x_3 goes to $+\infty$

$$u(x_3)\geqslant u(x_2)+\varepsilon(x_3-x_2),\quad \forall x_3>x_2.$$

Case 2. Consider now that $x_1>x_2$. Combining Lemma 1.4 and the fact that u is a subsolution for (7) yields

$$H(p) \leqslant H(0), \quad \forall p \in \left]u'(x_1), u'(x_2)\right[. \tag{18}$$

Assume that there is $x_3 < x_2$ such that $u'(x_3) < \varepsilon$. By Lemma 1.4 combined with the fact that u is a supersolution for (7) we obtain

$$H(p) \geqslant H(0), \quad p \in \left]u'(x_3), u'(x_2)\right[, \tag{19}$$

and therefore

$$H(p) = H(0), \quad p \in \left]\max\left\{u'(x_1), u'(x_3)\right\}, u'(x_2)\right[,$$

which is not possible in view of (16). Thus $u'(x_3) \geqslant \varepsilon$ for a.a. $x_3 < x_2$ leading to a contradiction since in this case $u(x_3)$ becomes unbounded when x_3 goes to $-\infty$

$$u(x_3) \leqslant u(x_2) - \varepsilon(x_2 - x_3), \quad \forall x_3 < x_2.$$

Once we have proved that u' has constant sign it is easily seen that every bounded solution for (7) is constant. Indeed if there is $x_1 \in \mathbb{R}$ such that $u'(x_1) > 0$ we know that $u'(x) > 0$ for a.a. $x \in \mathbb{R}$. Since $H(u'(x)) = H(0)$ for a.a. $x \in \mathbb{R}$ we deduce by (16) that $u'(x) \geqslant \varepsilon$ for a.a. $x \in \mathbb{R}$ and therefore u does not remain bounded. If there is $x_2 \in \mathbb{R}$ such that $u'(x_2) < 0$ we obtain a contradiction in a similar manner. Thus $u'(x) = 0$ for a.a. $x \in \mathbb{R}$ saying that u is constant. □

2. An application: explicit limiting solutions

The previous results enable us in certain cases to give explicitly the limiting solutions of initial value problems. We give here two examples.

Example 1. Consider the initial value problem

$$\begin{cases} \partial_t u + H(\partial_x u) = 0, & (x,t) \in \mathbb{R}\times]0,+\infty[, \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{20}$$

such that

$$H(0) = 0, \quad 0 \notin \overline{I_H - \{0\}}. \tag{21}$$

Then we have the following:

Proposition 2.1. *Let $H = H(p)$ satisfy* (5), (21)*, $u_0 \in W^{1,\infty}(\mathbb{R})$ and u be the solution of* (20).

(i) *If u_0 is a subsolution of $H(u_x) = 0$, $x \in \mathbb{R}$ then*

$$\lim_{t\to+\infty} u(x,t) = \sup_{y\in\mathbb{R}} u_0(y) =: \varphi_M, \quad \textit{uniformly for } x \textit{ in compact sets of } \mathbb{R}.$$

(ii) *If u_0 is a supersolution of $H(u_x) = 0$, $x \in \mathbb{R}$ then*

$$\lim_{t\to+\infty} u(x,t) = \inf_{y\in\mathbb{R}} u_0(y) =: \varphi_m, \quad \textit{uniformly for } x \textit{ in compact sets of } \mathbb{R}.$$

Proof. First notice that under the above assumptions, the problem (20) admits a unique bounded Lipschitz continuous solution. If u_0 is a subsolution of $H(u_x) = 0$, then one knows that $u(x,t)$ is nondecreasing in time and converges as t goes to infinity, uniformly for x in compact sets of $\mathbb{R}$, towards the minimal solution φ of $H(u_x) = 0$ which satisfies $\varphi(x) \geqslant u_0(x)$, $x \in \mathbb{R}$. But as (21) holds, Proposition 1.3 applies and therefore φ is necessarily a constant. Thus $\varphi(x) \equiv \varphi_M = \sup_{y\in\mathbb{R}} u_0(y)$. The second part (ii) follows in similar way. For results concerning long time behaviour of solutions of Hamilton–Jacobi equations, one can for example refer to the papers [1] and [6], see also [2]. □

Remark 1. Notice that the Hamiltonian of the above example can be quite general with no particular property of convexity or superlinearity type.

Now let us turn to an example with a periodic source term. Consider the initial value problem

$$\begin{cases} \partial_t u + \sqrt{1+(\partial_x u)^2} - 1 = \cos t, & (x,t) \in \mathbb{R}\times\,]0,+\infty[, \\ u(x,0) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{22}$$

Proposition 2.2. *For any $u_0 \in W^{1,\infty}(\mathbb{R})$ the solution of* (22) *verifies*

$$\lim_{t\to+\infty} \{u(x,t) - \sin t\} = \inf_{y\in\mathbb{R}} u_0(y), \quad \textit{uniformly for } x \textit{ in compact sets of } \mathbb{R}.$$

Proof. We will again use known results on the existence of time periodic solutions and on their asymptotic behaviour. We refer to [2] for example. First we know that (22) admits 2π periodic solutions. This comes from the solvability of $\sqrt{1+(u')^2} - 1 = (2\pi)^{-1}\int_0^{2\pi} \cos(t)\,\mathrm{d}t = 0,\ x \in \mathbb{R}$ (constants are solutions). Then observe that any $u_0 \in W^{1,\infty}(\mathbb{R})$ is a supersolution of $\sqrt{1+(u')^2} - 1 = 0$ so that

$$\lim_{t\to+\infty} \{u(x,t) - \sin t\} = \psi(x), \quad \text{uniformly for } x \text{ in compact sets of } \mathbb{R},$$

where ψ is the maximal solution of $\sqrt{1+(u')^2} - 1 = 0,\ x \in \mathbb{R}$ such that $\psi(x) \leqslant u_0(x),\ x \in \mathbb{R}$. Now as $0 \notin \overline{I_H - \{0\}} = \emptyset$, by Proposition 1.3 we deduce that ψ can only be a constant. The maximal constant ψ verifying $\psi(x) \leqslant u_0(x),\ x \in \mathbb{R}$ is necessarily given by the infimum of $\{u_0(y):\ y \in \mathbb{R}\}$. □

Remark 2. In fact in the above example we are just recovering known results for convex Hamiltonians, see [5] p. 251. Indeed, as in this case H is convex in p, one can obtain explicitly the solution via the Hopf–Lax formula applied to the equation satisfied by $v(x,t) = u(x,t) - \sin t$.

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