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On the preconditioned conjugate gradient solution of a Stokes problem with Robin-type boundary conditions

Roland Glowinski [a,b], Giovanna Guidoboni [b]

[a] *Laboratoire Jacques-Louis Lions, université P. et M. Curie, 4, place Jussieu, 75005 Paris, France*
[b] *Department of Mathematics, University of Houston, PGH 651, Houston, TX 77204-3476, USA*

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Abstract

We discuss the solution by a preconditioned conjugate gradient algorithm of a Stokes problem with Robin-type boundary conditions. This Stokes problem is encountered when applying an appropriate operator-splitting scheme to the time-discretization of a system modeling the interaction of an incompressible viscous fluid with a deformable thin elastic structure. The main contribution of this Note is the identification of a preconditioner operating in the pressure space that reduces substantially the number of iterations when compared to a conjugate gradient algorithm equipped with the canonical scalar product of L^2. The results of numerical experiments show the validity of our approach. ***To cite this article: R. Glowinski, G. Guidoboni, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Sur la résolution d'un problème de Stokes avec conditions aux limites du type Robin par un algorithme de gradient conjugué préconditionné. Dans cette Note on étudie la résolution d'un problème de Stokes avec conditions aux limites du type Robin par un algorithme de gradient conjugué préconditionné. On rencontre ce type de problèmes lorsque l'on applique certains schémas de discrétisation en temps par décomposition d'opérateurs à la résolution numérique du système d'équations aux derivées partielles modélisant l'interaction d'un fluide visqueux incompressible avec une structure élastique mince. La contribution principale de cette Note est l'identification d'un opérateur de préconditionnement agissant dans l'espace des pressions (en l'occurrence $L^2(\Omega)$) ; cet opérateur réduit considérablement le nombre d'itérations nécesssaires à la convergence, lorsque l'on compare à l'algorithme de gradient conjugué opérant dans $L^2(\Omega)$ muni de son produit scalaire canonique. Les résultats d'essais numériques confirment la validité de notre approche. ***Pour citer cet article : R. Glowinski, G. Guidoboni, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

E-mail addresses: roland@math.uh.edu (R. Glowinski), gio@math.uh.edu (G. Guidoboni).

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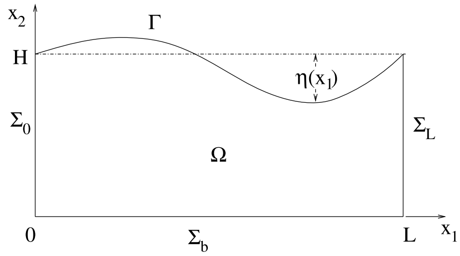

Fig. 1. Visualization of the flow region.

Fig. 1. Visualisation du domaine d'écoulement.

1. Formulation of the problem

Let Ω be a bounded domain of $\mathbb{R}^d$ filled with an incompressible viscous fluid of density ϱ_f and viscosity μ. For simplicity, in this Note we will consider only two-dimensional flows; the generalization of the method to three-dimensional situations is under investigation. We assume then that $\Omega \subset \mathbb{R}^2$ and that its boundary $\partial\Omega$ verifies $\partial\Omega = \Sigma_b \cup \Sigma_0 \cup \Sigma_L \cup \Gamma$ (see Fig. 1). We assume that there exists a smooth function $\eta = \eta(x_1)$ such that the upper boundary Γ can be written as follows

$$\Gamma = \big\{\{x_1, x_2\} \in \mathbb{R}^2\colon\ x_1 \in (0, L),\ x_2 = H + \eta(x_1)\big\}. \tag{1}$$

We prescribe the normal stress on Σ_0 and Σ_L, we assume symmetry with respect to the x_1-axis on Σ_b, and we assume that the top boundary Γ behaves like an elastic string. Such problem arises in many applications, including the modeling of blood flow in arteries (see e.g. [2–4,10]).

After an appropriate time-discretization by operator splitting which decouples the parabolic part of the problem (fluid viscosity) from the hyperbolic part (advection and elasticity) [8,9], one encounters at each time step a time-dependent Stokes problem of the following type:

$$\varrho_f \partial_t \mathbf{u} - \nabla\cdot\boldsymbol{\sigma} = \mathbf{0}, \quad \nabla\cdot\mathbf{u} = 0, \quad \text{in } \Omega \times (0, T) \tag{2}$$

with the initial condition $\mathbf{u}(0) = \mathbf{u}_0$, and the following boundary conditions:

$$\partial_{x_2} u_1 = 0 \quad \text{and} \quad u_2 = 0 \quad \text{on } \Sigma_b \times (0, T), \tag{3}$$

$$\boldsymbol{\sigma}\mathbf{n} = -\bar{p}\mathbf{n} \quad \text{on } \Sigma_0 \times (0, T), \quad \boldsymbol{\sigma}\mathbf{n} = \mathbf{0} \quad \text{on } \Sigma_L \times (0, T), \tag{4}$$

$$u_1 = 0 \quad \text{and} \quad \varrho_s h_s \partial_t u_2 + \sqrt{1 + (\partial_{x_1}\eta)^2}\,\boldsymbol{\sigma}\mathbf{n}\cdot\mathbf{e}_2 = 0 \quad \text{on } \Gamma \times (0, T), \tag{5}$$

where $\bar{p}$, ϱ_s and h_s are positive constants, and $\mathbf{e}_2 = (0, 1)$. Here $\mathbf{u} = \{u_1, u_2\}$ is the fluid velocity, p is the fluid pressure, and $\boldsymbol{\sigma}$ is the fluid stress tensor defined as $\boldsymbol{\sigma} = -p\mathbf{I} + 2\mu\mathbf{D}(\mathbf{u})$, with $\mathbf{D}(\mathbf{u}) = (\nabla\mathbf{u} + (\nabla\mathbf{u})^T)/2$. A variational formulation of problem (2)–(5) is: *For $t \in (0, T)$, find $\mathbf{u}(t) \in V$ and $p(t) \in L^2(\Omega)$ such that*:

$$\varrho_f \int_\Omega \partial_t\mathbf{u}\cdot\mathbf{v}\,d\mathbf{x} + \varrho_s h_s \int_0^L \partial_t(u_2|_\Gamma)v_2|_\Gamma\,dx_1 + 2\mu\int_\Omega \mathbf{D}(\mathbf{u}) : \mathbf{D}(\mathbf{v})\,d\mathbf{x} = \int_\Omega p\nabla\cdot\mathbf{v}\,d\mathbf{x} + \int_0^H \bar{p}v_1\,dx_2,$$
$$\int_\Omega q\nabla\cdot\mathbf{u} = 0, \quad \forall\mathbf{v} \in V \text{ and } \forall q \in L^2(\Omega), \text{ with } \mathbf{u}(0) = \mathbf{u}_0, \tag{6}$$

where $V = \{\mathbf{v} \in (H^1(\Omega))^2\colon\ v_2|_{\Sigma_b} = 0,\ v_1|_\Gamma = 0\}$. Let us discretize problem (6) by the backward Euler scheme. Then, with obvious notation, at each time step we have to solve: *Find $\mathbf{u}^n \in V$ and $p^n \in L^2(\Omega)$ such that*:

$$\alpha\int_\Omega \mathbf{u}^n\cdot\mathbf{v}\,d\mathbf{x} + \beta\int_0^L u_2^n|_\Gamma v_2|_\Gamma\,dx_1 + 2\mu\int_\Omega \mathbf{D}(\mathbf{u}^n) : \mathbf{D}(\mathbf{v})\,d\mathbf{x} = \int_\Omega p^n\nabla\cdot\mathbf{v}\,d\mathbf{x} + L_n(\mathbf{v}), \quad \forall\mathbf{v} \in V,$$

$$\int_{\Omega} q\nabla\cdot\mathbf{u}^n = 0, \quad \forall q \in L^2(\Omega), \tag{7}$$

where $\alpha = \varrho_f/\Delta t$, $\beta = \varrho_s h_s/\Delta t$, and $L_n(\mathbf{v}) = \alpha\int_{\Omega}\mathbf{u}^{n-1}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x} + \beta\int_0^L u_2^{n-1}|_{\Gamma} v_2|_{\Gamma}\,\mathrm{d}x_1 + \int_{\Sigma_0}\bar{p}v_1\,\mathrm{d}x_2$.

2. An equivalent pressure formulation of problem (7)

Let us define the following linear operator from $L^2(\Omega)$ into $L^2(\Omega)$

$$Aq = \nabla\cdot\mathbf{u}_q \quad \forall q \in L^2(\Omega), \tag{8}$$

where $\mathbf{u}_q$ is the unique solution of the following linear variational problem:

$$\mathbf{u}_q \in V; \quad \alpha\int_{\Omega}\mathbf{u}_q\cdot\mathbf{v}\,\mathrm{d}\mathbf{x} + \beta\int_0^L u_{q2}|_{\Gamma} v_2|_{\Gamma}\,\mathrm{d}x_1 + 2\mu\int_{\Omega}\mathbf{D}(\mathbf{u}_q):\mathbf{D}(\mathbf{v})\,\mathrm{d}\mathbf{x} = \int_{\Omega} q\nabla\cdot\mathbf{v}\,\mathrm{d}\mathbf{x}, \quad \forall\mathbf{v}\in V.$$

Following [6] Section 19.3, it can be shown that A is a strongly elliptic, symmetric automorphism of $L^2(\Omega)$. Thanks to these properties of operator A, problem (7) can be solved using a conjugate gradient method operating in $L^2(\Omega)$. Numerical experiments show that if the conjugate gradient method uses the classical scalar product in $L^2(\Omega)$, then the rate of convergence is pretty slow if $\alpha \gg \mu$. Unfortunately, this is the case for the application that motivates the present analysis, namely the modeling of blood flow in large arteries (see e.g. [2,3]). In order to improve the speed of convergence of the conjugate gradient algorithm, there is need for preconditioning (see [6] Chapter IV, Section 21.3). Preconditioning operators are available for the case of Dirichlet and/or stress related boundary conditions, but not for the boundary conditions in (5). This is the novelty introduced by the elasticity of Γ.

In order to avoid the deterioration of the convergence properties, associated with large values of the ratio α/μ, we will employ as scalar product on $L^2(\Omega)$ the one advocated in [5], namely

$$\{q,q'\} \to \int_{\Omega}\left(B^{-1}q\right)q'\,\mathrm{d}\mathbf{x} \quad \forall q,q' \in L^2(\Omega), \tag{9}$$

where the operator B^{-1} is defined via its inverse $B = \alpha(-\Delta)^{-1} + \mu I$; here I is the identity operator, and the Green operator $(-\Delta)^{-1}$ is associated with the boundary conditions (3)–(5) and it will be defined in the next section (see Eq. (17)). We remark that it is not necessary to know the operator B^{-1} explicitly, but it is sufficient to know its inverse B, see [6].

In order to identify the optimal preconditioning operator for the case at hand, we are going to consider a one dimensional situation where B can be computed explicitly.

3. A one dimensional case

Let us consider the bounded and open interval $(0,l)$. By analogy with the multi-dimensional case presented in Section 2, we define the operator A from $L^2(0,l)$ into $L^2(0,l)$ as $Aq = u_q'$, where u_q is the solution of the following problem

$$\alpha u_q - \mu u_q'' = -q' \quad \text{in } (0,l), \quad u_q(0) = 0, \quad \beta u_q(l) + \mu u_q'(l) = q(l). \tag{10}$$

Consider now the operator B defined by $Bq = \alpha\varphi_q + \mu q$ where φ_q is the solution of:

$$-\varphi_q'' = q \quad \text{in } (0,l), \quad \varphi_q'(0) = 0, \quad \varphi_q(l) + a\varphi_q'(l) = 0. \tag{11}$$

Theorem 3.1. *If $a = \beta/\alpha$, then $B^{-1} = A$.*

Proof. We need to show that $B(Aq) = Bu_q' = q$. By definition, $Bu_q' = \alpha\varphi_q + \mu u_q'$ where φ_q solves

$$-\varphi_q'' = u_q' \quad \text{in } (0,l), \quad \varphi_q'(0) = 0, \quad \varphi_q(l) + a\varphi_q'(l) = 0. \tag{12}$$

Therefore we have to show that $\alpha\varphi_q + \mu u_q' = q$. Integrating (12) from 0 to x and using the boundary conditions at $x = 0$ we obtain that

$$-\varphi_q'(x) = u_q(x), \tag{13}$$

which also holds at $x = l$. Now, combining (10) and (13) we get

$$-\alpha\varphi_q' - \mu u_q'' = -q'. \tag{14}$$

Integrating (14) from x to l, using (10), (11), and the fact that $-\varphi_q'(l) = u_q(l)$ we get:

$$a\alpha\varphi_q'(l) - \beta\varphi_q'(l) + \alpha\varphi_q(x) + \mu u_q'(x) = q(x). \tag{15}$$

This implies that if $a = \beta/\alpha$ then $\alpha\varphi_q + \mu u_q' = q$, and this completes the proof. □

4. A preconditioned conjugate gradient algorithm

Motivated by Theorem 3.1, we advocate the use of the following scalar product in $L^2(\Omega)$:

$$\{q, q'\} \to \int_\Omega (B^{-1}q)q' \,\mathrm{d}\mathbf{x} \tag{16}$$

with B defined as $Bq = \alpha\varphi_q + \mu q$, where φ_q solves the following problem:

$$-\nabla^2\varphi_q = q \quad \text{in } \Omega, \quad \varphi_q = 0 \quad \text{on } \Sigma_0 \cup \Sigma_L, \quad \partial\varphi_q/\partial n = 0 \quad \text{on } \Sigma_b, \quad \varphi_q + a\,\partial\varphi_q/\partial n = 0 \quad \text{on } \Gamma. \tag{17}$$

Finally, the preconditioned conjugate gradient algorithm to solve problem (7) reads as follows:

Take an initial guess $p^0 \in L^2(\Omega)$ and then find $\mathbf{u}^0 \in V$ such that

$$\alpha \int_\Omega \mathbf{u}^0 \cdot \mathbf{v}\,\mathrm{d}\mathbf{x} + \beta \int_0^L u_2^0|_\Gamma v_2|_\Gamma \,\mathrm{d}x_1 + 2\mu \int_\Omega \mathbf{D}(\mathbf{u}^0) : \mathbf{D}(\mathbf{v})\,\mathrm{d}\mathbf{x} = \int_\Omega p^0 \nabla \cdot \mathbf{v}\,\mathrm{d}\mathbf{x} + L(\mathbf{v}) \quad \forall \mathbf{v} \in V, \tag{18}$$

and set $r^0 = \nabla \cdot \mathbf{u}^0$. Solve now

$$-\nabla^2\varphi^0 = r^0 \quad \text{in } \Omega, \quad \varphi^0 = 0 \quad \text{on } \Sigma_0 \cup \Sigma_L, \quad \partial\varphi^0/\partial n = 0 \quad \text{on } \Sigma_b, \quad \varphi^0 + a\,\partial\varphi^0/\partial n = 0 \quad \text{on } \Gamma. \tag{19}$$

Then set $g^0 = \mu r^0 + \alpha\varphi^0$ and $w^0 = g^0$.
For $k \geqslant 0$, assuming that p^k, r^k, g^k, w^k are known, compute p^{k+1}, r^{k+1}, g^{k+1}, w^{k+1} as follows.
Find $\bar{\mathbf{u}}^k \in V$ such that

$$\alpha \int_\Omega \bar{\mathbf{u}}^k \cdot \mathbf{v}\,\mathrm{d}\mathbf{x} + \beta \int_0^L \bar{u}_2^k|_\Gamma v_2|_\Gamma \,\mathrm{d}x_1 + 2\mu \int_\Omega \mathbf{D}(\bar{\mathbf{u}}^k) : \mathbf{D}(\mathbf{v})\,\mathrm{d}\mathbf{x} = \int_\Omega w^k \nabla \cdot \mathbf{v}\,\mathrm{d}\mathbf{x} \quad \forall \mathbf{v} \in V, \tag{20}$$

and set $\bar{r}^k = \nabla \cdot \bar{\mathbf{u}}^k$. Compute

$$\varrho_k = \int_\Omega r^k g^k \,\mathrm{d}\mathbf{x} \Big/ \int_\Omega \bar{r}^k w^k \,\mathrm{d}\mathbf{x}, \tag{21}$$

and then $p^{k+1} = p^k - \varrho_k w^k$ and $r^{k+1} = r^k - \varrho_k \bar{r}^k$. Solve, next,

$$-\nabla^2\bar{\varphi}^k = \bar{r}^k \quad \textit{in } \Omega, \quad \bar{\varphi}^k = 0 \quad \textit{on } \Sigma_0 \cup \Sigma_L, \quad \partial\bar{\varphi}^k/\partial n = 0 \quad \textit{on } \Sigma_b, \quad \bar{\varphi}^k + a\,\partial\bar{\varphi}^k/\partial n = 0 \quad \textit{on } \Gamma. \tag{22}$$

Then compute $g^{k+1} = g^k - \varrho_k(\mu\bar{r}^k + \alpha\bar{\varphi}^k)$. If

$$\int_\Omega r^{k+1} g^{k+1} \,\mathrm{d}\mathbf{x} \Big/ \int_\Omega r^0 g^0 \,\mathrm{d}\mathbf{x} \leqslant \epsilon, \tag{23}$$

take $p = p^{k+1}$; else, compute $\gamma_k = \int_\Omega r^{k+1} g^{k+1} \,\mathrm{d}\mathbf{x} / \int_\Omega r^k g^k \,\mathrm{d}\mathbf{x}$, and update w^k by $w^{k+1} = g^{k+1} + \gamma_k w^k$.
Do $k = k + 1$ and return to (20).

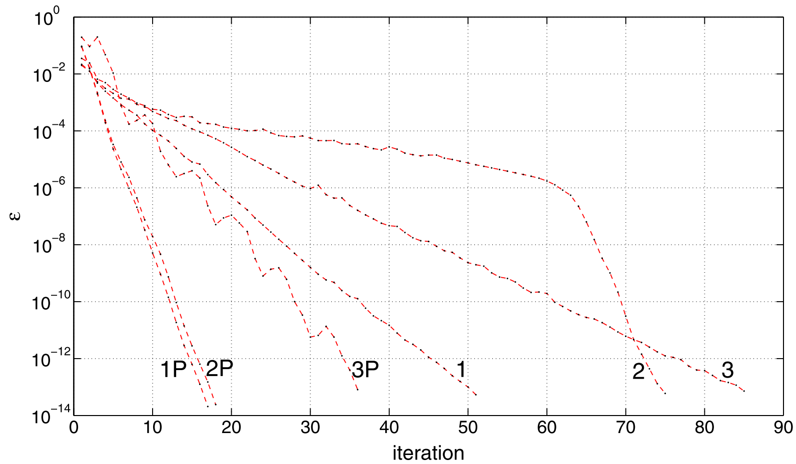

Fig. 2. Convergence of the conjugate gradient algorithm for *Case* 1, *Case* 2 and *Case* 3 with preconditioning (curves 1P, 2P and 3P) and without preconditioning (curves 1, 2, 3).

Fig. 2. Convergence de l'algorithme de gradient conjugué dans les *Cas* 1, 2 et 3 avec préconditionnement (courbes 1P, 2P et 3P) et sans préconditionnement (courbes 1, 2 et 3).

The main novelty of the scheme (18)–(23) consists in the new type of boundary conditions for the auxiliary function φ on the elastic portion of the boundary. From the classical theory of preconditioned conjugate gradient methods for incompressible viscous fluids (see [6] and the references therein), it is well known that where Dirichlet conditions for the normal component of the velocity are imposed, then one imposes $\partial\varphi/\partial n = 0$ for the auxiliary elliptic problem for the function φ. On the other hand, where a condition on the fluid stress is imposed, then one imposes $\varphi = 0$. For the boundary conditions of the problem at hand, we need to impose a Robin boundary condition on φ, see (19) and (22). The optimal value of the constant a in the Robin condition for φ is tightly related to the parameters α and β of problem (7). The optimality of this constant has been proved in Theorem 3.1 only for a simple one-dimensional problem, but our numerical experiments show the same trend in the multi-dimensional case.

5. Numerical results

To solve problem (18)–(23), we use an isoparametric version [6] of the Bercovier–Pironneau finite elements spaces [1]. This approximation, known as the $P1$-iso-$P2$ and $P1$ approximation, requires the use of two different meshes: a coarse mesh for the pressure (mesh size h_p) and a finer mesh for the velocity (mesh size $h_v = h_p/2$). Problems (18) and (20) are solved on the fine mesh, while problems (19) and (22) are solve on the coarse mesh. This significantly reduces the computational costs of the algorithm.

The numerical results presented in this section have been obtained for $H = 0.5$, $L = 6$, $\eta = 0$, $\mathbf{u}_0 = \mathbf{0}$, $p^0 = 0$, $h_p = H/8$, and $\epsilon = 10^{-13}$. We have considered three different cases:

- Case 1: $\alpha = 1 \times 10^3$, $\mu = 1.0$, $\beta = 1 \times 10^2$;
- Case 2: $\alpha = 1 \times 10^3$, $\mu = 1.0$, $\beta = 1 \times 10^4$;
- Case 3: $\alpha = 1 \times 10^3$, $\mu = 0.035$, $\beta = 1.1 \times 10^2$.

We remark that the parameter values encountered in blood flow modeling correspond to *Case* 3. In Fig. 2 we compare the performance of the preconditioned conjugate gradient algorithm with respect to its classical version (with the scalar product in L^2). In all the three cases presented here, the preconditioning significantly reduces the number of iterations needed for convergence. In Fig. 3 we show how the value of constant a in the Robin condition for φ (see (12), (19) and (22)) affects the speed of convergence of the preconditioned conjugate gradient scheme. The results in Fig. 3 correspond to *Case* 1, and they show that the optimal value for the constant a is around 0.1. This is in very satisfactory agreement with the optimal value predicted by Theorem 3.1, namely $a = \beta/\alpha = 0.1$. The results shown

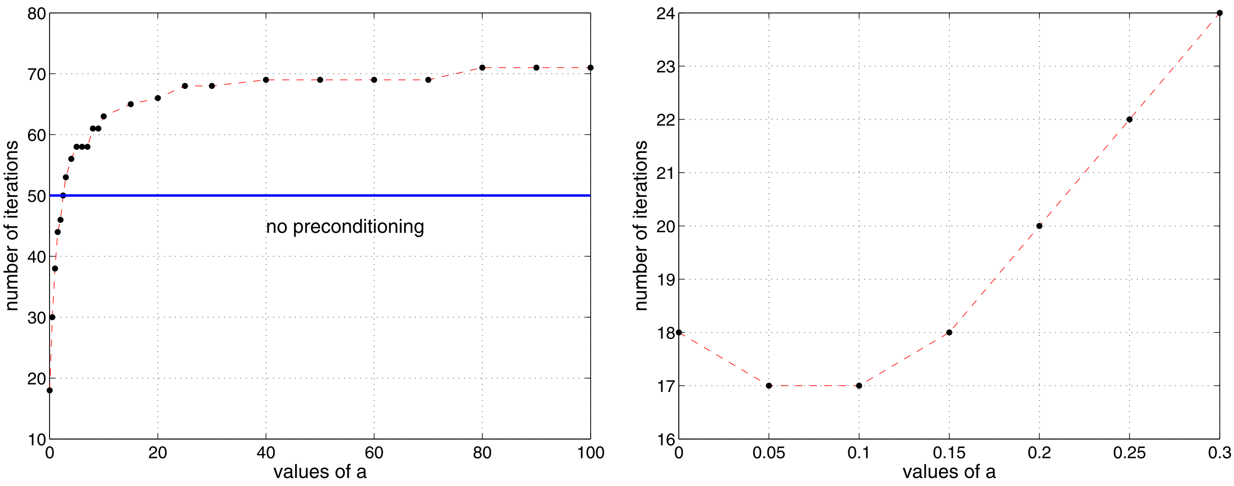

Fig. 3. Influence on the number of iterations of the constant a in the Robin condition verified by φ.

Fig. 3. Influence sur le nombre d'itérations de la constante a dans la condition de Robin vérifiée par φ.

in this Note are obtained for $\eta = 0$, but the case of $\eta = \eta(x_1)$ does not present any additional difficulty. Our approach can easily handle flows in domains with curved boundaries [8], even at high Reynolds numbers, see [7].

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