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COMPTES RENDUS MATHEMATIQUE

Partial Differential Equations

On the Boyd–Kadomtsev system for the three-wave coupling problem

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Abstract

The three-wave coupling system is widely used in plasma physics, specially for the Brillouin instability simulations. We study here a related system obtained with an infinite speed of light. After showing that it is well posed, we propose a numerical method which is based on an implicit time discretization. This method is illustrated on test cases and an extension to the problem with finite speed of light is proposed. ***To cite this article: R. Sentis, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***
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Résumé

Sur le système de Boyd–Kadomtsev pour le problème de couplage à trois ondes. Le système de couplage à trois ondes est très utilisé en physique des plasmas, particulierement pour les simulations des instabilités Brillouin. On étudie ici le système avec une vitesse de la lumière infinie ; on montre qu'il est bien posé. Une méthode de discrétisation implicite est proposée (extensible au cas de la vitesse de la lumière finie). ***Pour citer cet article : R. Sentis, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***
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Version française abrégée

Le système de couplage à trois ondes a été l'objet de nombreuses études à partir des années 60 [3,1] pour les problèmes d'interaction d'ondes dans les plasmas. Après normalisation, on cherche (U, V, W) dépendant du temps et d'une variable monodimensionelle x (appartenant à $[0, L]$) satisfaisant (1)–(3). A notre connaissance, à part [7] (recherche de solitons avec des techniques valables seulement sur $\mathbf{R}$ entier), il n'a pas été publié de travail mathématiquement satisfaisant sur ce système. D'après [5], on sait qu'il est bien posé dans $L^\infty_{\mathrm{loc}}(0, +\infty; L^\infty(O, L))$. Il est très utilisé dans le cadre l'interaction laser-plasma, U et V correspondent alors à l'onde laser incidente et à l'onde laser rétrodiffusée (Brillouin) et W à l'onde accoustique ionique. Notons que pour traiter de façon réaliste l'instabilité Brillouin, il convient de se placer dans un cadre tri-dimensionnel et de rajouter au système précédent des termes pour prendre en compte la diffraction, la réfraction et les effets hydrodynamiques du plasma (cf. [2,6,8,4,9]). Cependant, les difficultés majeures tant sur le plan mathématique que numérique peuvent être vues sur le système précédent.

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La motivation de cette note est de proposer une méthode numérique avec un pas de temps δt qui n'est pas contraint par le rapport du pas d'espace δx sur la vitesse de la lumière. Dans ce but, nous étudions le système de Boyd–Kadomtsev asymptotique (5), (6), obtenu en faisant $\beta = 0$.

• L'Analyse du système de Boyd–Kadomtsev asymptotique est abordée en section 2 avec le résultat principal suivant :

Théorème 2.1. *Supposons que* $W_{\text{ini}} \in L^\infty(0, L)$; *alors pour tout* T_f, *il existe une unique solution faible* (U, V, W) *dans* $[C^0(0, T_f, L^2(0, L))]^3$ *pour le système* (5), (6). *Elle vérifie de plus* $|V(t, 0)| \leqslant 1$.

Précisément, W vérife (12) où l'opérateur Λ est défini par $\Lambda(W) = u\bar{v}$ avec (u, v) est solution de (7).

• En section 3, on aborde la discrétisation temporelle et on propose un schéma numérique. A chaque pas de temps n de longueur $\delta t = 2h$, on opère un splitting en deux étapes.

i. A partir de $W^{(n)}$, on obtient $\widetilde{W}$ en résolvant (18) par un schéma explicite en temps.

ii. On pose $W^{(n+1)} = \widetilde{W} + 2h\gamma\Lambda_h(\widetilde{W})$ sachant que $\Lambda_h(\widetilde{W}) = u\bar{v}$ avec (u, v) solution de (20). Pour cela, grâce à une méthode itérative, on détermine μ et $z = u\bar{v}$, solution de (21). Puis, avec les expressions de $|u|$, $|v|$ données par (14), on résout successivement les deux EDO (23). On a la stabilité dans L^2 grâce à

Proposition 3.2. *Si* $W^{(n)} \in L^\infty(0, L)$, *alors* $W^{(n+1)}$ *est bornée et vérifie* (22).

• Dans la section 4, ce schéma est illustré par des résultats numériques. On propose aussi une adaptation au système de Boyd–Kadomtsev complet.

1. Introduction

To model the wave interaction in plasmas, [3] and [1] have addressed the following system:

$$(\beta\partial_t + \partial_x)U = -\gamma V W, \tag{1}$$

$$(\beta\partial_t - \partial_x)V = \gamma U\overline{W}, \tag{2}$$

$$(\partial_t + \partial_x + \eta + i\omega)W = \gamma U\overline{V} \tag{3}$$

supplemented with initial and boundary conditions, where the three complex functions U, V, W depend on the non-dimension time and space variables t and x (here $x \in [0, L]$). In the framework of the laser-plasma interaction, U, V, W correspond to the incoming laser wave, the backscattered laser wave due to the Brillouin instability and the ion acoustic wave. The real constant numbers β, η and ω are related to the ratio between the sound speed and the light speed, the Landau damping and the wave number. The real function γ such that $0 < \gamma \leqslant 1$ is related to the profile of the plasma density. We get easily the following energy balance relations.

$$\beta\partial_t\big(|U|^2 + |V|^2\big) + \partial_z\big(|U|^2 - |V|^2\big) = 0, \qquad \partial_t\big(\beta|U|^2 + |W|^2\big) + 2\eta|W|^2 + \partial_z\big(|W|^2 + |U|^2\big) = 0 \tag{4}$$

(the first one is related to the laser energy). Up to our knowledge, except to the work [7] on solitons (on the full space) there is no convincing published mathematical work related to this system. For two different velocities c_1, c_2, denote $\mathcal{M}_k = \{u = u(t, x) \mid (\partial_t + c_k\partial_x)u \in L^2_{t,x},\ u|_{t=0} \in L^2_x,\ u|_{x=0} \in L^2_t\}$. Using a compensated integrability result which claims that there exists C such that

$$\|uv\|^2_{L^2_{t,x}} \leqslant C\big(B_u + \big\|(\partial_t + c_1\partial_x)u\big\|^2_{L^2_{t,x}}\big)\big(B_v + \big\|(\partial_t + c_2\partial_x)v\big\|^2_{L^2_{t,x}}\big) \quad \text{with } B_u = \|u|_{t=0}\|^2_{L^2_x} + \|u|_{x=0}\|^2_{L^2_t}$$

for all $u \in \mathcal{M}_1$, $v \in \mathcal{M}_2$ and using bounds of U, V, W, in $L^\infty_{t,x}$ (see [5]), one first checks that the above system is well-posed in $L^\infty \cap L^2([0, \tau] \times [0, L])$ for τ small enough and afterwards for all τ.

Of course when dealing with realistic simulations one has to address three-dimension geometry and to account for diffraction, refraction phenomena as well as macroscopic hydrodynamic effects (see [2,6,8,4] for such models and [9] for mathematical justifications); but this Boyd–Kadomtsev system is sufficient to exhibit most of the difficulties of the three-wave coupling. Since the typical value of β is in the order of 10^{-3}, we only consider in the sequel the problem after neglecting the terms $\beta\partial_t$, i.e. it reads

$$\partial_x U = -\gamma V W, \qquad -\partial_x V = \gamma U \overline{W}, \tag{5}$$

$$(\partial_t + \partial_x + \eta + i\omega) W = \gamma U \overline{V}. \tag{6}$$

The interesting problem corresponds to the following boundary conditions

$$U(t,0) = U^{\text{in}}, \quad V(t,L) = 0, \quad W(t,0) = 0, \quad \forall t.$$

It may be assumed that $U^{\text{in}} = 1$. The initial condition is $W(0,\cdot) = W_{\text{ini}}$ belonging to $L^\infty(0,L)$. Notice that if $W_{\text{ini}} = 0$, there exists a trivial solution: $V(t,\cdot) = W(t,\cdot) = 0$ and $U(t,\cdot) = 1$.

2. Analysis of the asymptotic Boyd–Kadomtsev system

Theorem 2.1. *Assume that* $W_{\text{ini}} \in L^\infty(0,L)$; *then for all* T_f, *there exists a unique weak solution* (U, V, W) *in* $[C^0(0,T_f,L^2(0,L))]^3$ *of the system* (5), (6). *Moreover we get* $|V(t,0)| \leqslant 1$.

This section is devoted to its proof, but first the meaning of weak solution has to be made more precise.

Lemma 2.2. *For all function* $W \in L^\infty(0,L)$, *there exists a unique bounded solution* (u,v) *to*

$$\partial_x u = -\gamma W v, \quad u|_{x=0} = 1, \quad \text{and} \quad -\partial_x v = \gamma \overline{W} u, \qquad v|_{x=L} = 0. \tag{7}$$

Let us denote $\Lambda(W) = u\bar{v}$. *Moreover, there exists a constant* $C(w_\infty)$ *depending on* w_∞ *such that*

$$2\,\mathrm{Re}\big(\langle \gamma \Lambda(W), W\rangle\big) \leqslant 1, \qquad \big|\Lambda(W)(0)\big| = \big|v(0)\big| \leqslant 1. \tag{8}$$

$$\big\|\Lambda(W)\big\|_\infty \leqslant A^2, \quad \|u\|_\infty \leqslant A, \quad \|v\|_\infty \leqslant A, \quad \text{with } A = \exp\big(\sqrt{L}\|W\|_{L^2}\big), \tag{9}$$

$$\big\|\Lambda(W) - \Lambda(W')\big\|_\infty \leqslant C(w_\infty)\|W - W'\|_{L^2}, \quad \forall W, W', \quad \|W\|_\infty, \|W'\|_\infty \leqslant w_\infty. \tag{10}$$

($\langle , \rangle$ denotes the scalar product in $L^2(0,L)$.) Thanks to this lemma, the original system reads as

$$(\partial_t + \partial_x + \eta + i\omega) W = \gamma \Lambda(W), \quad W(t,\cdot)|_{x=0} = 0, \quad W(0,\cdot) = W_{\text{ini}}. \tag{11}$$

Let $\mathbf{T}_t$ be the semi-group $\exp(-(\partial_x + \eta + i\omega)t)$. So U, V, W is a weak solution means that

$$W(t) = \mathbf{T}_t W^{\text{ini}} + \int_0^t \mathbf{T}_{t-s}\big[\gamma \Lambda\big(W(s)\big)\big]\,\mathrm{d}s. \tag{12}$$

Sketch of the proof of the lemma. *A priori estimates*. If (u,v) satisfy (7), we get

$$\partial_x |u|^2 = -2\gamma\,\mathrm{Re}(W\bar{u}v), \qquad -\partial_x |v|^2 = 2\gamma\,\mathrm{Re}(W\bar{u}v) \tag{13}$$

and $\partial_x(|u|^2 - |v|^2) = 0$. Thus there exists a constant μ satisfying $|u|^2 - |v|^2 = \mu$. Since $|u|(0) = 1$, we get $0 < \mu \leqslant 1$. Moreover, due to $(|u|^2 + |v|^2)^2 - \mu^2 = 4|z|^2$, we get

$$|u|^2 = \sqrt{|z|^2 + \mu^2/4} + \mu/2, \qquad |v|^2 = \sqrt{|z|^2 + \mu^2/4} - \mu/2. \tag{14}$$

Integrating (13) over $[0,L]$, we get $\int \gamma 2\,\mathrm{Re}(\overline{W}z) = -\int \partial_x |u|^2 \leqslant 1$. Moreover (13) implies $-\partial_x(|u|^2 + |v|^2) \leqslant 2|W|(|u|^2 + |v|^2)$. Thus (9) follows. We check that $z = u\bar{v}$ solves the ODE

$$-\partial_x z = \gamma W \sqrt{4|z|^2 + \mu^2}, \quad z(L) = 0. \tag{15}$$

Existence. We have to find $\mu > 0$ and z solution to (15) satisfying $|z(0)|^2 = 1 - \mu$. It suffices to take $Z = z\frac{2}{\mu}$ which solves $-\partial_x Z = \gamma W \sqrt{|Z|^2 + 1}$ and $\mu = 2(\sqrt{|Z(0)|^2 + 1} - 1)|Z(0)|^{-2}$. □

Sketch of the proof of the theorem. *A priori estimates.* Assume that W is solution of (11). Set $\|W_{\text{ini}}\|_\infty = \alpha$, then according to (8), we get $\partial_t \|W(t)\|_{L^2}^2 + \eta \|W(t)\|_{L^2}^2 \leqslant 1$ and $\|W(t)\|_{L^2} \leqslant \alpha + T_f$, for all $t \leq T_f$. Now, using (9), $\Lambda(W)$ is bounded in L^∞ and due to the property of the semi-group $\mathbf{T}_t$ we get

$$\big\|W(t)\big\|_\infty \leqslant w_\infty, \quad \text{with } w_\infty = \alpha + T_f \exp\big(2\sqrt{L}(\alpha + T_f)\big). \tag{16}$$

Uniqueness. It comes from Gronwall's lemma and the inequality (due to (10))

$$\|(W'-W)(t)\|_{L^2} \leqslant \sqrt{L}C(w_\infty)\int_0^t \|(W'-W)(s)\|_{L^2}\,\mathrm{d}s. \tag{17}$$

Existence. Build the sequence $(\partial_t + \partial_x + \eta + i\omega)W^{n+1} = \gamma\Lambda(W^n)$, then check that it converges in $C^0(0,t_f; L^2(0,L))$ for t_f small enough. So the time uniform estimate on $\|W\|_{L^2}$ yields the result. □

3. Time discretization and numerical scheme

We propose a time discretization of system (5), (6) such that the energy balance relations (4) [with $\beta = 0$] are satisfied at discrete level. Denote $\delta t = 2h$ the time step and $W^{(n)}$ the value at time $t = n\delta t$. We first evaluate the intermediate value $\widetilde{W} = W(\delta t)$ by solving on $[0, \delta t)$

$$\partial_t W + \partial_x W + (\eta + i\omega)W = 0, \quad \text{with initial value} \quad W(0) = W^{(n)}. \tag{18}$$

Then define $W^{(n+1)}$ by $\widetilde{W} + 2h\gamma u\bar{v}$ where u, v solve (5) with $W = \frac{1}{2}(W^{(n+1)} + \widetilde{W})$, i.e.

$$W^{(n+1)} = \widetilde{W} + 2h\gamma\Lambda_h(\widetilde{W}) \tag{19}$$

where $\Lambda_h(\widetilde{W}) = u\bar{v}$ knowing that (u, v) is solution to

$$\partial_x u + \gamma^2 hu|v|^2 = -\gamma\widetilde{W}v, \quad u(0) = U^{\text{in}} \quad \text{and} \quad -\partial_x v - \gamma^2 hv|u|^2 = \gamma\overline{\widetilde{W}}u, \quad v(L) = 0. \tag{20}$$

Proposition 3.1. *Let $\widetilde{W} \in L^\infty(0,L)$. There exists a bounded solution (u, v) of* (20). *We have $\int 2\gamma\,\mathrm{Re}(\overline{\widetilde{W}}\Lambda(\widetilde{W})) \leqslant 1$ and $|v(0)| < 1$. Moreover $\|\Lambda_h(\widetilde{W})\|_\infty \leqslant A^2$, $\|u\|_\infty \leqslant A$, $\|v\|_\infty \leqslant A$ (A given by* (9)).

The key point is that (20) is equivalent to find $z = u\bar{v}$ and μ such that

$$-\partial_x z = \gamma^2 hz\sqrt{4|z|^2 + \mu^2} + \gamma W\sqrt{4|z|^2 + \mu^2}, \quad |z(0)| = \sqrt{1-\mu}, \quad z(L) = 0. \tag{21}$$

Proposition 3.2. *If $W^{(n)} \in L^\infty(0,L)$, then $W^{(n+1)}$ is also bounded and satisfies the following estimate*:

$$\|W^{(n+1)}\|_{L^2}^2 - \|W^{(n)}\|_{L^2}^2 + h|u(L)|^2 \leqslant h. \tag{22}$$

So the time discretization is stable. The quantities h and $h|u(L)|^2$ in (22) correspond to the incoming and outgoing laser energy. From a practical point of view, the scheme is the following:

Stage 1. Solve (18) by a standard upwind scheme to get a value for $\widetilde{W}$; for stability, dealing with the advection operator implies a CFL stability condition (here we simply take $\delta x = \delta t$).

Stage 2. The aim is to solve system (20).

a) First, determine μ and z solution of (21) by an iterative method.

b) Then, using these quantities and the expressions of $|u|$ and $|v|$ given by (14), solve the two ODEs.

$$\partial_x u = -\gamma^2 hu|v|^2 - \gamma\widetilde{W}\bar{z}\frac{u}{|u|^2}, \quad u(0) = 1 \quad \text{and} \quad -\partial_x v = \gamma^2 hv|u|^2 + \gamma\overline{\widetilde{W}}u, \quad v(L) = 0. \tag{23}$$

Lastly, the value of $W^{(n+1)}$ at the end of the time step is given by (19).

4. Numerical results. Extension

4.1. Numerical results

We address asymptotic Boyd–Kadomtsev system on an interval $[0, L]$ with $L = 10$ and the initial data W_{ini} is a smooth real function such that $\alpha = 0.05$. Here $\gamma = 1$. We plot on Fig. 1 and Fig. 2 the profiles of $|U|^2$, $|V|^2$ and $|W|^2$ versus x at different times for two cases: $\eta = 0.1$ and $\eta = 1$ (here we have $\delta x = \delta t = 0.05$). We also plot on Fig. 3 the

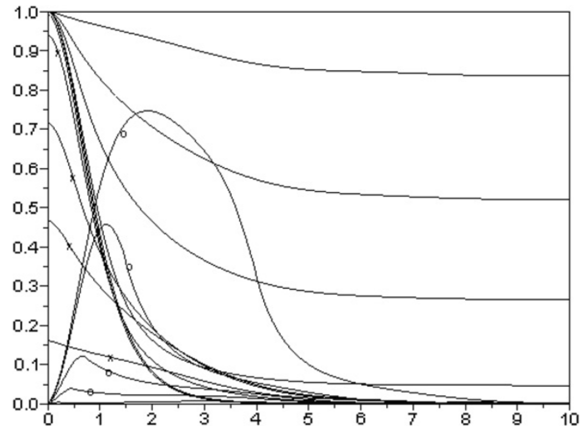

Fig. 1. Profiles of $|U|^2$, (without mark) $|V|^2$, (with a '×') and $|W|^2$ (with a 'o') for $\eta = 0.1$ at different time values.

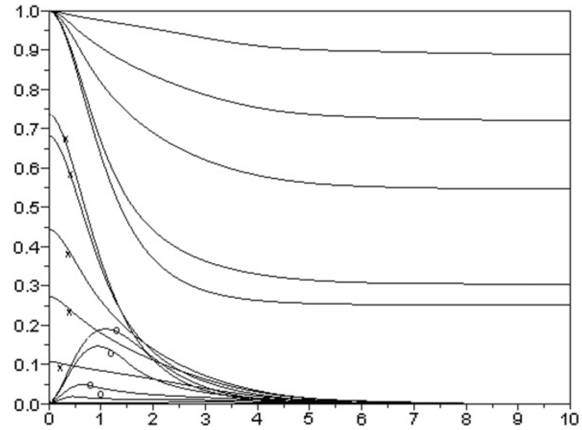

Fig. 2. Profiles of $|U|^2$, (without mark) $|V|^2$, (with a '×') and $|W|^2$ (with a 'o') for $\eta = 1$ at different time values.

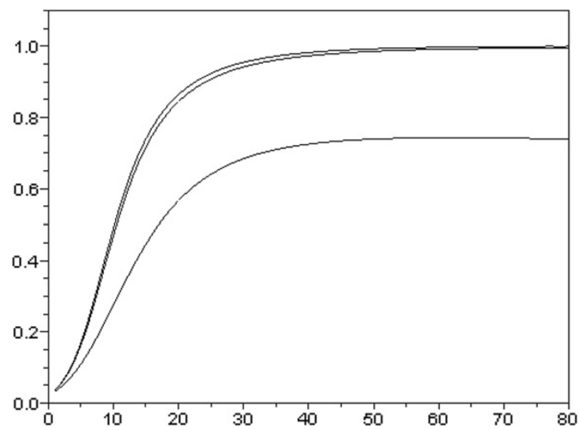

Fig. 3. Time evolution of the backscattered energy $r(t)$ for different values of η (from bottom to top $\eta = 1$, 0.1 and 0.01).

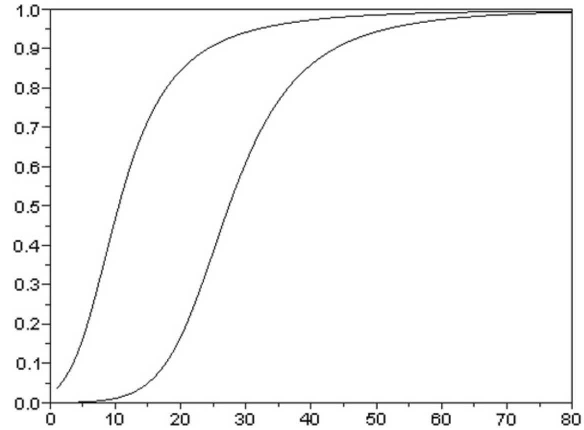

Fig. 4. Time evolution of the backscattered energy $r(t)$ for two initial values W_{ini} (for smaller I.V. the curve is shifted to the right).

results of time evolution of the backscattered energy $r(t) = |V(t,0)|^2$ for three values of the coefficient η ($\eta = 0.01$, 0.1 and 1). The behavior of the function $r(t)$ is strongly modified when the damping coefficient η becomes larger. In Fig. 4, we show the results with the same data and $\eta = 0.1$ but with a much smaller initial value W_{ini} (it is divided by 10). Notice that the time evolution of the curve $r(t)$ is shifted with respect to t, but the maximum value is the same.

4.2. Extension

Let us address the full Boyd–Kadomtsev system. At each time step, we perform stages 1 and 2-a) as above; this gives the values of $z, |u|, |v|$. With these values and obvious notations u^{pre} and v^{pre}, stage 2-b) consists in solving

$$\frac{\beta}{2h}u + \partial_x u + \gamma^2 h u|v|^2 + \gamma \widetilde{W}\bar{z}\frac{u}{|u|^2} = \frac{\beta}{2h}u^{\text{pre}}, \qquad \frac{\beta}{2h}v - \partial_x v - \gamma^2 h v|u|^2 - \gamma \overline{\widetilde{W}} u = \frac{\beta}{2h}v^{\text{pre}}.$$

Due to the value of β, the terms $\frac{\beta}{2h}$ are only corrective ones (here $h \approx 0.02$ but $\beta \approx 10^{-3}$).

To conclude, notice that it is possible that some difficulties have to be overcome when applying this numerical method to three-dimension simulations where diffraction phenomena have to be taken into account; but the proposed numerical method seems to be a good alternative for such simulations with respect to the classical ones where the time step is determined by the space step divided by the speed of light.

References

[1] T.J.M. Boyd, J.G. Turner, Lagrangian studies of plasma wave interaction, J. Phys. A: Gen. Phys. 5 (1972) 881–896.
[2] V.V. Eliseev, W. Rozmus, V.T. Tikhonchuk, C.E. Capjack, On the Brillouin scattering, Phys. Plasmas 2 (1995) 1712.
[3] B.B. Kadomtsev, Plasma Turbulence, Academic Press, 1965. Translated from the Russian of a book edited in 1964.
[4] P. Loiseau, et al., Laser beam smoothing induced by stimulated . . . , Phys. Rev. Lett. 97 (2006) 205001.
[5] G. Metivier, J. Rauch, Brillouin backscattering, Private communication.
[6] Ph. Mounaix, D. Pesme, M. Casanova, Nonlinear reflectivity of an inhomogeous plasma, Phys. Rev. E 55 (1997) 4653.
[7] S. Novikov, V.E. Zakharov, et al., Theory of Solitons, Consultant Bureau, New York, 1984.
[8] D. Pesme, Interaction collisionnelle et collective, in: R. Dautray, M. Watteau (Eds.), La fusion thermonucléaire inertielle I. Interaction laser-matière, Eyrolles, Paris, 1995 (Chapitre 2).
[9] R. Sentis, Mathematical models for laser-plasma, M2AN Math. Model. Numer. Anal. 39 (2005) 275–318.