

Partial Differential Equations

Existence and concentration for nonlinear Schrödinger equations with fast decaying potentials

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Abstract

The existence of positive solutions to

$$-\varepsilon^2 \Delta u + Vu = u^p \quad \text{in } \mathbb{R}^N,$$

is proved for small ε , where $N \geq 3$, V is a nonnegative continuous potential which has a positive local minimum, and $\frac{N}{N-2} < p < \frac{N+2}{N-2}$. No restriction is imposed on the rate of decay of V ; this includes in particular the case where V is compactly supported. **To cite this article:** V. Moroz, J. Van Schaftingen, C. R. Acad. Sci. Paris, Ser. I 347 (2009).

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Résumé

Existence et concentration pour l'équation de Schrödinger non linéaire avec des potentiels à décroissance rapide. Nous prouvons l'existence de solutions positives non triviales de

$$-\varepsilon^2 \Delta u + Vu = u^p \quad \text{dans } \mathbb{R}^N,$$

pour ε petit, où $N \geq 3$, V est potentiel continu positif qui a un minimum local strictement positif et $\frac{N}{N-2} < p < \frac{N+2}{N-2}$. Nous n'imposons aucune restriction sur le taux de décroissance de V . En particulier, nous couvrons le cas où le support de V est compact.

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On considère l'équation de Schrödinger non linéaire (1). On sait que quand V atteint un minimum global strictement positif non dégénéré en $x_0 \in \mathbb{R}^N$ cette équation admet des solutions qui se concentrent autour de x_0 [7]. Ce travail a été étendu dans différentes directions dans le cas où $\inf_{\mathbb{R}^n} V > 0$ [2].

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Nous nous intéressons au cas où $V > 0$ mais $\inf_{\mathbb{R}^N} V = 0$ dont l'étude a été initiée par A. Ambrosetti, V. Felli et A. Malchiodi [1] et par A. Ambrosetti, A. Malchiodi and D. Ruiz [3] qui ont établi l'existence sous la condition de décroissance lente (3). D. Bonheure et J. Van Schaftingen ont dans le cas $\frac{N}{N-2} < p < \frac{N+2}{N-2}$, affaibli cette condition en (4).

Nous annonçons dans cette Note l'existence et la concentration de solutions de (1) dans le cas de potentiels à décroissance rapide (5) qui incluent en particulier les potentiels à support compact. Rappelons que si le support de V est compact, (1) n'a pas de solution dans un voisinage de l'infini si $p \leq \frac{N}{N-2}$ [8]. Nous obtenons

Théorème 0.1. *Soit $N \geq 3$, $\frac{N}{N-2} < p < \frac{N+2}{N-2}$ et $V \in C(\mathbb{R}^N, \mathbb{R}^+)$. Supposons qu'il existe un ouvert régulier $\Lambda \subset \mathbb{R}^N$ tel que $0 < \inf_{\Lambda} V < \inf_{\partial\Lambda} V$. Alors, pour tout $\varepsilon > 0$ suffisamment petit, l'équation (1) a une solution positive non triviale u_ε .*

Les méthodes s'appliquent en fait au problème plus général (6) où $N \geq 2$, $p > 1$, $\varepsilon > 0$ et $V, K \in C(\mathbb{R}^N, \mathbb{R}^+)$. Les points de concentration sont des points critiques de la fonction de concentration \mathcal{A} définie en (7) et l'existence sera obtenue dans l'espace de Sobolev à poids défini en (8).

Théorème 0.2. *Soit $N \geq 2$, $1 < p < 2^*$ et soit, $K \in C(\mathbb{R}^N, \mathbb{R}^+)$. Supposons qu'il existe $\sigma < (N-2)p - N$ et $M > 0$ tels que (9) ait lieu et qu'il existe un ouvert borné régulier $\Lambda \subset \mathbb{R}^N$ tel que $0 < \inf_{\Lambda} \mathcal{A} < \inf_{\partial\Lambda} \mathcal{A}$. Alors il existe $\varepsilon_0 > 0$ tel que pour tout $\varepsilon \in (0, \varepsilon_0)$, (6) ait au moins une solution positive non triviale $u_\varepsilon \in \mathcal{D}_V^1(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)$. De plus, $\|u_\varepsilon\|_\varepsilon = O(\varepsilon^{N/2})$ quand $\varepsilon \rightarrow 0$, u_ε atteint son maximum en $x_\varepsilon \in \Lambda$, x_ε satisfait (10) et il existe $C, \lambda > 0$ tels que pour tout $\varepsilon \in (0, \varepsilon_0)$, u_ε a la décroissance (11).*

Si le support de V est compact, la restriction $\sigma < (N-2)p - N$ dans le théorème est optimale, dans le sens que (6) n'a pas de solutions positives pour $\sigma \geq (N-2)p - N$ [8], et la borne supérieure (11) est optimale dans le sens que $\liminf_{|x| \rightarrow \infty} |x|^{N-2} u_\varepsilon(x) > 0$, pour tout $\varepsilon \in (0, \varepsilon_0)$.

La preuve suit la méthode de pénalisation de M. del Pino et P. Felmer [6] adaptée ensuite par D. Bonheure et J. Van Schaftingen [5]. Le Théorème 0.2 est valable pour $N \geq 2$, mais la preuve du cas $N = 2$ est différente de celle du cas $N \geq 3$, sur lequel nous nous concentrons dans la suite.

On définit un potentiel de pénalisation H défini par (12) où $\beta > 0$, $0 < \rho_0 < \rho$, $B(0, \rho) \subset \Lambda$ et $0 < \frac{\kappa}{\log(\rho/\rho_0)} < \frac{(N-2)^2}{4}$. On vérifie entre autres que $-\Delta - H$ satisfait un principe de comparaison entre sur- et sous-solutions et qu'il possède des sursolutions qui décroissent comme $|x|^{-(N-2)}$ à l'infini. On définit ensuite par (14) une non-linéarité pénalisée, et on montre que la fonctionnelle associée (15) a un point critique qui satisfait l'équation (16).

Il reste alors à montrer que u_ε satisfait l'équation non pénalisée (6). Pour ce faire on commence par étudier le comportement asymptotique local de u_ε . On montre que (17) a lieu et que si $\varepsilon_n \rightarrow 0$, $(x_n) \subset \Lambda$, $\liminf_{n \rightarrow \infty} u_{\varepsilon_n}(x_n) > 0$ et $x_n \rightarrow x_0 \in \Lambda$, alors $\mathcal{A}(x_0) = \inf_{\Lambda} \mathcal{A}$, et à une sous-suite près la suite $v_n(x) := u_{\varepsilon_n}(x_n + \varepsilon_n x)$ converge vers une solution de (18). On en déduit qu'il existe $R > 0$ tel que pour $\varepsilon > 0$ u_ε satisfait (19). On considère ensuite une sur-solution w_ε de (19) définie par (20) où $\mu = \frac{1}{2} \sqrt{\inf_{\Lambda} V}$ et $\tilde{w} \in C^2(\mathbb{R}^N)$ satisfait $-\Delta \tilde{w} - H \tilde{w} = 0$ en dehors de Λ et vaut 1 sur un voisinage de la famille (x_ε) . Par le principe de comparaison et par (17), pour $\varepsilon > 0$ petit, on conclut que $u_\varepsilon \leq w_\varepsilon$ in $B(x_\varepsilon, r)^c$. On en déduit u_ε a la décroissance (11), d'où par l'hypothèse (9), u_ε satisfait (21). Par construction de g_ε , u_ε est une solution du problème initial (1).

1. Introduction

We are interested in a nonlinear Schrödinger equations of the type

$$-\varepsilon^2 \Delta u + V u = u^p \quad \text{in } \mathbb{R}^N, \quad (1)$$

where $\varepsilon > 0$ is a small parameter, $N \geq 3$, $V \in C(\mathbb{R}^N, \mathbb{R}^+)$, and $1 < p < \frac{N+2}{N-2}$.

In the case where V achieves a nondegenerate positive global minimum at $x_0 \in \mathbb{R}^N$ it is known that for all small $\varepsilon > 0$ Eq. (1) admits positive solutions $u_\varepsilon \in H^1(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)$, such that $\|u_\varepsilon\|_{H^1(\mathbb{R}^N)} \rightarrow 0$ as $\varepsilon \rightarrow 0$, and the family (u_ε) concentrates around x_0 : there exists $x_\varepsilon \in \mathbb{R}^N$

$$\liminf_{\varepsilon \rightarrow 0} u_\varepsilon(x_\varepsilon) > 0 \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} \max_K u_\varepsilon(x) = 0 \quad \text{for every compact } K \subset \mathbb{R}^N \setminus \{x_0\},$$

where $x_0 = \lim_{\varepsilon \rightarrow 0} x_\varepsilon$. Moreover, when $\varepsilon \rightarrow 0$ the rescaled functions $v_\varepsilon(x) := u_\varepsilon(x_0 + \varepsilon x)$ converge in $C^1_{\text{loc}}(\mathbb{R}^N)$ to the unique positive radial solution $v_0 > 0$ of the limiting equation

$$-\Delta v_0 + V(x_0)v_0 = v_0^p \quad \text{in } \mathbb{R}^N. \tag{2}$$

This result goes back to the pioneering work by A. Floer and A. Weinstein [7], and has been extended in different directions (see e.g. [2] for an extensive list of references). In most of these works the assumption $\inf_{\mathbb{R}^N} V > 0$ was made.

We are interested in the case where $V > 0$ but $\inf_{\mathbb{R}^N} V = 0$, whose study was initiated by A. Ambrosetti, V. Felli and A. Malchiodi in [1] and by A. Ambrosetti, A. Malchiodi and D. Ruiz [3]. In particular, in [3] the existence and concentration of solutions to (1) was proved when V satisfies the *slow decay* assumption

$$\liminf_{|x| \rightarrow \infty} V(x)|x|^2 > 0. \tag{3}$$

D. Bonheure and J. Van Schaftingen [4,5] established the existence and concentration of solution to (1) when

$$\liminf_{|x| \rightarrow \infty} V(x)|x|^{(N-2)(p-1)} > 0, \tag{4}$$

which thus provides an improvement to the results in [3] when $p > \frac{N}{N-2}$.

In this Note, we announce the existence and concentration of solutions to (1) in the case of *fast decaying potentials*, i.e. potentials for which

$$\liminf_{|x| \rightarrow \infty} V(x)|x|^2 = 0. \tag{5}$$

This includes, in particular, potentials with compact support in \mathbb{R}^N .

One issue in the study of equations (1) with fast decaying potentials is that its solutions may have slow polynomial decay as $x \rightarrow \infty$. Thus, any concentration result should capture a transition between polynomial decay of the concentrating solutions u_ε and exponential decay of the limiting solution that is obtained by rescaling the problem. Another issue is that polynomial decay of solutions brings Liouville-type nonexistence phenomena, which restrict the admissible exponents. For example, if V is compactly supported, Eq. (1) has no positive solutions in the neighborhood of infinity for $p \leq \frac{N}{N-2}$, cf. [8].

A special case of our results is the following:

Theorem 1.1. *Let $N \geq 3$, $\frac{N}{N-2} < p < \frac{N+2}{N-2}$ and $V \in C(\mathbb{R}^N, \mathbb{R})$ be a nonnegative potential. Assume that there exists a smooth bounded open set $\Lambda \subset \mathbb{R}^N$ such that $0 < \inf_\Lambda V < \inf_{\partial\Lambda} V$. Then for small $\varepsilon > 0$ Eq. (1) has a positive solution u_ε .*

2. The main result

Theorem 1.1 follows from results for the slightly more general problem

$$-\varepsilon^2 \Delta u + Vu = Ku^p \quad \text{in } \mathbb{R}^N, \tag{6}$$

where $N \geq 2$, $p > 1$, $\varepsilon > 0$ and $V, K \in C(\mathbb{R}^N, \mathbb{R}^+)$. The existence of solutions will be related to the local minimizers of the concentration function

$$\mathcal{A}(x) := V(x)^{\frac{p+1}{p-1} - \frac{N}{2}} K(x)^{-\frac{2}{p-1}}, \tag{7}$$

and will be obtained in the weighted Sobolev space induced by the linear part of the equation,

$$\mathcal{D}_V^1(\mathbb{R}^N) := \left\{ u \in \mathcal{D}_0^1(\mathbb{R}^N) \mid \|u\|_\varepsilon = \int_{\mathbb{R}^N} (\varepsilon^2 |\nabla u|^2 + V|u|^2) < +\infty \right\}. \tag{8}$$

Here $\mathcal{D}_0^1(\mathbb{R}^N)$ is the closure of $C_c^\infty(\mathbb{R}^N)$ with respect to the L^2 -norm of the gradient.

Theorem 2.1. *Let $N \geq 2$, $1 < p < 2^*$ and let $V, K \in C(\mathbb{R}^N, \mathbb{R}^+)$. Assume that there exists $\sigma < (N - 2)p - N$ and $M > 0$ such that*

$$0 \leq K(x) \leq M(1 + |x|)^\sigma \quad \text{for all } x \in \mathbb{R}^N, \tag{9}$$

and that there exists a smooth bounded open set $\Lambda \subset \mathbb{R}^N$ such that $0 < \inf_\Lambda \mathcal{A} < \inf_{\partial\Lambda} \mathcal{A}$. Then there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$, Eq. (6) has at least one positive solution $u_\varepsilon \in \mathcal{D}_V^1(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)$. Moreover, $\|u_\varepsilon\|_\varepsilon = O(\varepsilon^{N/2})$ as $\varepsilon \rightarrow 0$, u_ε attains its maximum at $x_\varepsilon \in \Lambda$,

$$\liminf_{\varepsilon \rightarrow 0} u_\varepsilon(x_\varepsilon) > 0, \quad \lim_{\varepsilon \rightarrow 0} \mathcal{A}(x_\varepsilon) = \inf_{x \in \Lambda} \mathcal{A}(x), \tag{10}$$

and there exists $C, \lambda > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ holds

$$u_\varepsilon(x) \leq C \exp\left(-\frac{\lambda}{\varepsilon} \frac{|x - x_\varepsilon|}{1 + |x - x_\varepsilon|}\right) (1 + |x - x_\varepsilon|^2)^{-\frac{N-2}{2}}. \tag{11}$$

If V has compact support then the restriction $\sigma < (N - 2)p - N$ in the theorem is sharp, in the sense that (6) has no positive solutions for $\sigma \geq (N - 2)p - N$ (see e.g. [8]), and the upper bound (11) is sharp in the sense that $\liminf_{|x| \rightarrow \infty} |x|^{N-2} u_\varepsilon(x) > 0$, for each fixed $\varepsilon \in (0, \varepsilon_0]$.

3. Sketch of the proofs

The proof of Theorem 2.1 and its extensions will appear in [9]. The approach is based on an adequate modification of the penalization scheme of M. del Pino and P. Felmer [6], subsequently adapted by D. Bonheure and J. Van Schaftingen [5]. Whereas Theorem 2.1 holds for $N \geq 2$, there are substantial differences in the proof between the case $N = 2$ and the case $N \geq 3$. We sketch only the case $N \geq 3$.

3.1. The penalization potential

Assuming without loss of generality that $0 \in \Lambda$ and that $\overline{B(0, \rho)} \subset \Lambda$, define $H : \mathbb{R}^N \rightarrow \mathbb{R}$ by

$$H(x) := \frac{\kappa(1 - \chi_\Lambda(x))}{|x|^2(\log(|x|/\rho_0))^{1+\beta}}, \tag{12}$$

where χ_Λ is the characteristic function of Λ , $\beta > 0$, $0 < \rho_0$, and $0 < \frac{\kappa}{\log(\rho/\rho_0)} < \frac{(N-2)^2}{4}$. This potential has the following properties

- (i) by the classical Hardy inequality, the quadratic form associated to $-\Delta - H$ on \mathbb{R}^N is positive;
- (ii) $-\Delta - H$ satisfies a comparison principle between sub- and super-solutions;
- (iii) the minimal positive solution of

$$-\Delta w - Hw = 0 \quad \text{in } \bar{\Lambda}^c, \quad w = 1 \quad \text{on } \partial\Lambda, \tag{13}$$

satisfies $c|x|^{-(N-2)} \leq w(x) \leq C|x|^{-(N-2)}$.

3.2. The penalized problem

The truncated nonlinearity $g_\varepsilon : \mathbb{R}^N \times \mathbb{R}^+ \rightarrow \mathbb{R}$ is defined by

$$g_\varepsilon(x, s) := \chi_\Lambda(x)K(x)s^p + \min(\varepsilon^2 H(x), K(x)s^{p-1})s, \tag{14}$$

and $G_\varepsilon(x, s) := \int_0^s g_\varepsilon(x, t) dt$. The penalized functional is

$$\mathcal{J}_\varepsilon(u) := \frac{1}{2} \int_{\mathbb{R}^N} (\varepsilon^2 |\nabla u(x)|^2 + V(x)|u(x)|^2) dx - \frac{1}{p+1} \int_{\mathbb{R}^N} G_\varepsilon(x, u(x)) dx. \tag{15}$$

One can verify that $\mathcal{J}_\varepsilon \in C^1(\mathcal{D}_V^1(\mathbb{R}^N), \mathbb{R})$, satisfies the Palais–Smale condition (cf. [5, Lemmas 5,6]) and has the mountain-pass geometry at 0. Hence the mountain-pass lemma implies the existence of a family of positive solutions $(u_\varepsilon) \subset \mathcal{D}_V^1(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)$ to the penalized equation

$$-\varepsilon^2 \Delta u + V(x)u = g_\varepsilon(x, u) \quad \text{in } \mathbb{R}^N. \tag{16}$$

It remains to establish that (u_ε) solve the original problem (1) and has the required asymptotic behavior.

3.3. Asymptotics of the solutions

By the comparison principle, we first prove that $\liminf_{\varepsilon \rightarrow 0} \|u_\varepsilon\|_{L^\infty(\Lambda)} > 0$. One then establishes as in [5, Proposition 16], that if the family $(x_\varepsilon) \subset \Lambda$ satisfies $\liminf_{n \rightarrow \infty} u_{\varepsilon_n}(x_\varepsilon) > 0$ then $\liminf_{\varepsilon \rightarrow 0} d(x_\varepsilon, \partial\Omega) > 0$ and

$$\lim_{\substack{\varepsilon \rightarrow 0 \\ R \rightarrow \infty}} \|u_\varepsilon\|_{L^\infty(\Lambda \setminus B(x_\varepsilon, \varepsilon R))} = 0. \tag{17}$$

Moreover, if $\varepsilon_n \rightarrow 0$, and $(x_n) \subset \Lambda$ is a sequence such that $\liminf_{n \rightarrow \infty} u_{\varepsilon_n}(x_n) > 0$ and $x_n \rightarrow x_0 \in \Lambda$, then $\mathcal{A}(x_0) = \inf_\Lambda \mathcal{A}$, and, up to a subsequence a subsequence, $v_n(x) := u_{\varepsilon_n}(x_n + \varepsilon x)$ converges in $C_{\text{loc}}^1(\mathbb{R}^N)$ to a positive solution $v \in H^1(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)$ of the limiting equation

$$-\Delta v + V(x_0)v = K(x_0)v^p \quad \text{in } \mathbb{R}^N. \tag{18}$$

This local information will be crucial to obtain sharp global decay estimates on the solutions from which we will deduce that the solutions solve the original problem (1).

3.4. Barrier functions and solutions of the original problem

By construction of the penalization, u_ε is a subsolution to $-\varepsilon^2 \Delta - \varepsilon^2 H + V$ in Λ^c . By the comparison principle argument and by (17) there exists $R > 0$ so that for all small $\varepsilon > 0$ one has

$$-\varepsilon^2 \Delta u_\varepsilon - \varepsilon^2 H u_\varepsilon + \frac{1}{2} V u_\varepsilon \leq 0 \quad \text{in } \mathbb{R}^N \setminus B(x_\varepsilon, \varepsilon R). \tag{19}$$

Let $w \in C^2(\Lambda^c)$ be the minimal positive solution of (13). Set $r = \frac{1}{3} \liminf_{\varepsilon \rightarrow 0} d(x_\varepsilon, \partial\Lambda)$, and let $\tilde{w} \in C^2(\mathbb{R}^N)$ be a positive extension of w such that $\tilde{w}(x) = 1$ if $d(x, \Lambda^c) \geq r$. Define a family of barrier functions w_ε by

$$w_\varepsilon(x) = \begin{cases} \cosh \frac{\mu(r-|x|)}{\varepsilon} & \text{if } x \in B(x_\varepsilon, r), \\ \tilde{w}(x) & \text{if } x \in B(x_\varepsilon, r)^c, \end{cases} \tag{20}$$

where $\mu = \frac{1}{2} \sqrt{\inf_\Lambda V}$. A direct computation shows that w_ε is a supersolution to (19). By the comparison principle and (17), for small $\varepsilon > 0$ one concludes that $u_\varepsilon \leq w_\varepsilon$ in $B(x_\varepsilon, r)^c$. This, in particular, implies the sharp concentration bound (11).

Finally, using assumption (9) and (11), for small $\varepsilon > 0$ and all $x \in \Lambda^c$ we obtain

$$K(x)(u_\varepsilon(x))^{p-1} \leq M(1+|x|)^\sigma (C e^{-\frac{\lambda}{\varepsilon}|x|} |x|^{-(N-2)})^{p-1} \leq C M e^{-\frac{\lambda}{\varepsilon}} (1+|x|)^{-(N-2)(p-1)+\sigma} \leq \varepsilon^2 H(x). \tag{21}$$

By the construction of g_ε , one has then $g_\varepsilon(x, u_\varepsilon(x)) = K(x)(u_\varepsilon(x))^p$ for all $x \in \mathbb{R}^N$, and therefore u_ε solves the original problem (1).

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