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Partial Differential Equations

Trapping Rossby waves

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Abstract

Waves associated to large scale oceanic motions are gravity waves (Poincaré waves which disperse fast) and quasigeostrophic waves (Rossby waves). In this Note, we show by semiclassical arguments, that Rossby waves can be trapped and we characterize the corresponding initial conditions. ***To cite this article: C. Cheverry et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Ondes de Rossby piégées. Les ondes associées aux mouvements océaniques à grande échelle sont les ondes de gravité (dites de Poincaré) qui dispersent très vite, et les ondes quasigéostrophiques (dites de Rossby). Dans cette Note, nous montrons par une analyse semiclassique que les ondes de Rossby peuvent être piégées et nous caractérisons les conditions initiales correspondantes.

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Version française abrégée

Le mouvement des océans à grande échelle peut être décrit – en première approximation – par les équations linéarisées de Saint-Venant pour les couches minces, avec force de Coriolis, i.e. par le système bidimensionnel (1) où B est la composante verticale locale du vecteur rotation de la Terre (qui ne dépend que de la latitude x_2), ρ représente la fluctuation de hauteur d'eau, et u le champ de vitesses horizontal. Pour simplifier, on suppose que $(x_1, x_2) \in \mathbb{R} \times \mathbb{T}$, ce qui signifie qu'on néglige l'influence des bords latéraux.

Les ondes associées à ce système linéaire sont usuellement regroupées en deux familles : les ondes de Poincaré ou ondes de gravité qui dispersent très vite, et les ondes de Rossby beaucoup plus lentes liées aux inhomogénéités de B.

Dans cette Note, nous nous intéressons, dans la limite de grandes valeurs de la force B, à la propagation des ondes de Rossby et montrons que, conformément aux observations expérimentales, ces ondes peuvent être piégées et donner

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lieu à des zones, dites de ventilation, non influencées par la dynamique et les sources extérieures, en particulier par la présence des continents.

Plus précisément, en considérant $B = b/\epsilon$ où ϵ est le nombre de Rossby (ainsi que la longueur d'onde caractéristique du mouvement, liée au forçage par le vent, voir [1]), nous montrons (Théorème 1.1) qu'il existe une sous-variété Λ de $T^*(\mathbb{R} \times \mathbb{T})$ telle que, si la partie Rossby de la condition initiale est microlocalisée hors de Λ, alors la solution « sort » (modulo ϵ^∞) de tout compact en temps de l'ordre de ϵ^{-1}. Si au contraire, une partie du mode de Rossby de la condition initiale est contenue dans $\Lambda \cap T^*\Omega$ pour Ω compact de $\mathbb{R} \times \mathbb{T}$, alors, pour tout temps de l'ordre de ϵ^{-1}, la norme $L^2(\Omega)$ de la solution n'est pas $O(\epsilon^\infty)$. Dans le cas d'une condition initiale de type WKB, le résultat est plus précis. On montre que si la variété lagrangienne, graphe de la fonction de phase, intersecte Λ, la norme $L^2(\Omega)$ de la solution à temps $\frac{t}{\epsilon}$ est d'ordre $C(t)$ avec $C(t) > 0$ pour tout t.

La preuve se décompose en plusieurs étapes : on montre tout d'abord que l'on peut se ramener à une situation scalaire grâce à trois opérateurs pseudo-différentiels vectoriels $\mathbb{P}^+$, $\mathbb{P}^-$, $\mathbb{P}^0$ (définis en fin de Section 2) qui découplent les trois modes (deux de Poincaré et un de Rossby). Un argument spectral, possible en raison de la petite dimension du problème (conditions de Bohr–Sommerfeld) permet ensuite de montrer la dispersion des ondes de Poincaré en temps $O(\epsilon^{-1})$ dès que b^2 n'a pas de point critique dégénéré. Pour le mode de Rossby, on a alors une équation d'évolution (2.1) scalaire à résoudre, dont les bicaractéristiques s'avèrent être périodiques pour certaines conditions initiales, définissant justement Λ (Section 3). On conclut par un argument de propagation microlocale.

1. Introduction and results

Large scale oceanic motions can be described – at first sight – by the linearized Saint-Venant equations for thin layers, with Coriolis force, i.e. by the following two-dimensional system [6]:

$$\partial_t U + \begin{pmatrix} 0 & \partial_{x_1} & \partial_{x_2} \\ \partial_{x_1} & 0 & -B(x_2) \\ \partial_{x_2} & B(x_2) & 0 \end{pmatrix} U = 0, \quad \text{where } U = U(t, x_1, x_2) = \begin{pmatrix} \rho \\ u_1 \\ u_2 \end{pmatrix} \tag{1}$$

where B is the local vertical component of the Earth rotation vector (depending only on the latitude x_2), ρ denotes the fluctuation of water height and u the horizontal velocity field. For the sake of simplicity, we assume that $(x_1, x_2) \in \mathbb{R} \times \mathbb{T}$, meaning that we neglect the influence of the lateral boundaries.

Waves associated to that linear system are usually classified in two families: First, Poincaré waves, which are fast dispersive gravity waves; Second, Rossby waves, due to the inhomogeneities of B, which propagate much more slowly [2,3].

In this Note, we are interested in the propagation of Rossby waves. We show, in agreement with physical observations [7], that these waves can be trapped in some regions, called ventilation zones, which are not influenced by external dynamics and sources such as continental recirculation for instance.

More precisely, in the limit of large values of $B = b/\epsilon$, where ϵ is the Rossby number as well as the typical wavelength (corresponding to wind forcing, see [1]), we construct a co-dimension 1 submanifold Λ of $T^*(\mathbb{R} \times \mathbb{T})$. This set contains the ϵ-microlocalization region inside which, essentially, an initial condition of (1) remains trapped. Let us first recall that the ϵ-frequency set of a function u [4] is the (closed) subset of $T^*(\mathbb{R} \times \mathbb{T})$, complement of the set of points $(\underline{x}, \underline{\xi})$ such that there exists a C^∞ function χ, with $\chi(\underline{x}, \underline{\xi}) = 1$, such that

$$\left\| \int \chi\left(\frac{x+y}{2}, \xi\right) e^{i\frac{\xi(x-y)}{\epsilon}} u(y)\,\mathrm{d}y\,\mathrm{d}\xi \right\|_{L^2} = O\big(\epsilon^\infty\big).$$

Theorem 1.1. *Let us consider the system*

$$\partial_t U + \begin{pmatrix} 0 & \partial_{x_1} & \partial_{x_2} \\ \partial_{x_1} & 0 & -\frac{b(x_2)}{\epsilon} \\ \partial_{x_2} & \frac{b(x_2)}{\epsilon} & 0 \end{pmatrix} U = 0 \quad \textit{with an initial condition } U|_{t=0} = \begin{pmatrix} \rho^0 \\ u_1^0 \\ u_2^0 \end{pmatrix} \tag{2}$$

for some smooth b such that b^2 has no degenerate critical point, with $L^2(\mathbb{R} \times \mathbb{T}, \mathrm{d}x_1\,\mathrm{d}x_2)$ conditions (periodic in x_2). Let $\Lambda = \{F(\xi_1, x_2, \xi_2) = 0\} \subset T^*\mathbb{R} \times \mathbb{T}$, where F is defined in Lemma 3.1.

We suppose that the ϵ-frequency set of $U|_{t=0}$ is contained in a compact set $\mathcal{C}$ satisfying:

$$\mathcal{C} \cap \{\xi_1 = 0\} = \mathcal{C} \cap \{\xi_2^2 + b(x_2)^2 = 0\} = \emptyset. \tag{3}$$

Let us fix a compact set Ω of $\mathbb{R} \times \mathbb{T}$. Then there exists ($\epsilon$-)pseudo-differential operators P_ρ^0, P_1^0, P_2^0 of principal symbols $p_\rho^0 = ib(x_2)\xi_1(\xi_2^2 + \xi_1^2 + b^2(x_2))^{-1}$, $p_1^0 = -\xi_1\xi_2(\xi_2^2 + \xi_1^2 + b^2(x_2))^{-1}$, $p_2^0 = \xi_1^2(\xi_2^2 + \xi_1^2 + b^2(x_2))^{-1}$, such that:

(i) *if the ϵ-frequency set of $P_\rho^0\rho^0 + P_1^0u_1^0 + P_2^0u_2^0$ does not intersect $\Lambda \cap T^*\Omega$, then $\exists T > 0$ s.t.*

$$\forall t \geqslant T, \quad \left\| U\left(\frac{t}{\epsilon}\right) \right\|_{L^2(\Omega)} = \mathrm{O}(\epsilon^\infty);$$

(ii) *if the ϵ-frequency set of $P_\rho\rho^0 + P_1u_1^0 + P_2u_2^0$ does intersect $\Lambda \cap T^*\Omega$, then $\exists\Omega'$ compact, $\Omega \subset \Omega'$, such that $\forall t \geqslant 0$, $\|U(\frac{t}{\epsilon})\|_{L^2(\Omega')} \neq \mathrm{O}(\epsilon^\infty)$ (in other words the frequency set of $U(\frac{t}{\epsilon})$ intersects $T^*\Omega'$).*

In the case of a WKB initial condition the conclusion is more precise. Consider

$$U|_{t=0} = \begin{pmatrix} R^0(x) \\ U_1^0(x) \\ U_2^0(x) \end{pmatrix} e^{i\frac{S(x)}{\epsilon}}.$$

Suppose than the (Lagrangian) manifold $\{(x, \nabla S(x)), (p_\rho^0R^0 + p_1^0U_1^0 + p_2^0U_2^0)(x, \nabla S(x)) \neq 0\}$ intersects $\Lambda \cap T^*\Omega$, then:

$$\left\| U\left(\frac{t}{\epsilon}\right) \right\|_{L^2(\Omega')} \sim C(t) + \mathrm{O}(\epsilon), \quad C(t) > 0. \tag{4}$$

2. Reduction to a scalar situation

Performing first a Fourier analysis in x_1 (as the system does not contain explicitly this variable) and looking secondly at modes of the system (2), we can prove the following proposition, theheart of our results:

Proposition 2.1. *There exist three pseudo-differential operators $T_\pm$, T_0, of leading symbols*

$$\tau_\pm = \pm\sqrt{\xi_1^2 + \xi_2^2 + b^2(x_2)}, \qquad \tau_0 = \epsilon\frac{b'(x_2)\xi_1}{\xi_1^2 + \xi_2^2 + b^2(x_2)},$$

roots of (A.1)*, such that, if $u_{2,j}^0$ is microlocalized outside $\xi_1^2 - \tau_j^2 = 0$, we have, for $j \in \{-, 0, +\}$:*

$$\epsilon\partial_t u_{2,j} = iT_ju_{2,j} \Longrightarrow U_j := \begin{pmatrix} (i\epsilon\partial_{x_2}T_j + \epsilon\partial_{x_1}b(x_2))(\epsilon^2\partial_{x_1}^2 + T_j^2)^{-1} \\ -(\epsilon^2\partial_{x_1}\partial_{x_2} + ib(x_2)T_j)(\epsilon^2\partial_{x_1}^2 + T_j^2)^{-1} \\ \mathbb{I}d \end{pmatrix} u_{2,j} \text{ satisfies (2) up to } \mathrm{O}(\epsilon^\infty).$$

The proof is a consequence of the result contained in Appendix A.

The following result shows that any initial condition of (2) can be decomposed on the three modes of the last section:

Proposition 2.2. $\forall\rho, u_1, u_2$, $\exists u_{2,j}$, $j \in \{+, 0, -\}$ *such that:*

$$\begin{pmatrix} \rho \\ u_1 \\ u_2 \end{pmatrix} = \sum_j \begin{pmatrix} c(i\epsilon\partial_{x_2}T_j + \epsilon\partial_{x_1}b(x_2))(\epsilon^2\partial_{x_1}^2 + T_j^2)^{-1} \\ -(\epsilon^2\partial_{x_1}\partial_{x_2} + ib(x_2)T_j)(\epsilon^2\partial_{x_1}^2 + T_j^2)^{-1} \\ \mathbb{I}d \end{pmatrix} u_{2,j} + \mathrm{O}(\epsilon^\infty) =: \sum_j \mathbb{Q}^j u_{2,j} + \mathrm{O}(\epsilon^\infty). \tag{5}$$

To prove the proposition one has just to invert the matrix: $(\mathbb{Q}^- \, \mathbb{Q}^0 \, \mathbb{Q}^+)$. Semiclassically it is enough to show that the matrix of the leading order symbol is invertible

$$\begin{pmatrix} \frac{\xi_2\sqrt{\xi_1^2+\xi_2^2+b(x_2)^2}+i\xi_1 b(x_2)}{\xi_2^2+b(x_2)^2} & -\frac{ib(x_2)}{\xi_1} & \frac{-\xi_2\sqrt{\xi_1^2+\xi_2^2+b(x_2)^2}+i\xi_1 b(x_2)}{\xi_2^2+b(x_2)^2} \\ \frac{\xi_1\xi_2+ib(x_2)\sqrt{\xi_1^2+\xi_2^2+b(x_2)^2}}{\xi_2^2+b(x_2)^2} & -\frac{\xi_2}{\xi_1} & \frac{\xi_1\xi_2-ib(x_2)\sqrt{\xi_1^2+\xi_2^2+b(x_2)^2}}{\xi_2^2+b(x_2)^2} \\ 1 & 1 & 1 \end{pmatrix}. \tag{6}$$

A simple computation shows that the Jacobian $J = \frac{2(\xi_1^2+\xi_2^2+b^2(x_2))^{3/2}}{(\xi_2^2+b^2(x_2))|\xi_1|} \geqslant 2$. In particular the inversion of the matrix $(\mathbb{Q}^- \, \mathbb{Q}^0 \, \mathbb{Q}^+)$ can be done symbolically at any order and gives the leading order symbols of the operators $\mathbb{P}^j := (P_\rho^j, P_1^j, P_2^j)$ such that $u_{2,j} = P_\rho^j \rho + P_1^j u_1 + P_2^j u_2$, $j \in \{-, 0, +\}$. One gets: $p_\rho^0 = ib(x_2)\xi_1(\xi_2^2 + \xi_1^2 + b^2(x_2))^{-1}$, $p_1^0 = -\xi_1\xi_2(\xi_2^2 + \xi_1^2 + b^2(x_2))^{-1}$, $p_2^0 = \xi_1^2(\xi_2^2 + \xi_1^2 + b^2(x_2))^{-1}$.

3. Propagation under the Rossby Hamiltonian τ_0

Thanks to Proposition 2.1 it is enough, as far as the Rossby mode is concerned, to look at propagation with respect to the Hamiltonian τ_0/ϵ. This Hamiltonian being independent of x_1, ξ_1 will be conserved. The flow is periodic in the variables x_2, ξ_2 (one degree of freedom). Therefore, since $\dot{x}_1 = \frac{b'(x_2)(\xi_2^2-\xi_1^2+b^2(x_2))}{(\xi_2^2+\xi_1^2+b^2(x_2))^2}$ is periodic (the case of infinite and zero period is treated in [1]), $x_1(t)$ will contain a part, linear in time except if

$$F\big(\xi_1, x_2(0), \xi_2(0)\big) := \int_0^{period} \frac{b'(x_2(t))(\xi_2(t)^2 - \xi_1^2 + b^2(x_2(t)))}{(\xi_2(t)^2 + \xi_1^2 + b^2(x_2(t)))^2} \, \mathrm{d}t = 0.$$

Lemma 3.1. *As $b'(x_2) \neq 0$, $b'(x_2)F(\xi_1, x_2, \xi_2) > 0$ as $\xi_1 \to \pm\infty$, < 0 as $\xi_1 \to 0$ and is invariant under the flow of $\frac{\tau_0}{\epsilon}$. Define $E(\xi_1, x_2, \xi_2) = \frac{b'(x_2)\xi_1}{\xi_2^2+\xi_1^2+b^2(x_2)}$. Then*

$$\big|F(\xi_1, x_2, \xi_2)\big| = \left| \int_{x_-}^{x_+} \frac{\frac{b'(x)}{E(\xi_1,x_2,\xi_2)} - 2\xi_1}{\sqrt{\frac{b'(x)\xi_1}{E(\xi_1,x_2,\xi_2)} - \xi_1^2 - b^2(x)}} \, \mathrm{d}x \right|,$$

where $]x_-, x_+[$ is the largest interval of $\mathbb{T}$ containing x_2 in which $\frac{b'(x)\xi_1}{E(\xi_1,x_2,\xi_2)} - \xi_1^2 - b^2(x) > 0$.

We define $\Lambda := \{(x_1, x_2; \xi_1, \xi_2) \mid F(\xi_1, x_2, \xi_2) = 0\}$. Thanks to Lemma 3.1 $\Lambda \neq \emptyset$ and $\dim \Lambda = 3$.

Corollary 3.2. *Suppose $b'(x_2) \neq 0$. Then $|F(\xi_1, x_2, \xi_2)| \geqslant \frac{C(x_2,\xi_2)}{\xi_1}$ as $\xi_1 \to 0$, with $C(x_2, \xi_2) > 0$. This implies that the trapping phenomenon will take place only with initial conditions oscillating enough in x_1.*

Remarks. 1. In the case when $b'(x_2) \neq 0$, the trapped trajectories are not reduced to fixed points and generate non-trivial macroscopic structures, described in [1].

2. Since the Hamiltonian E does not depend on x_1, we can express it, for each value of ξ_1, on the action variable A: $E(\xi_1, x_2, \xi_2) = H(A, \xi_1)$. This allows to define the function $A(\xi_1, x_2, \xi_2)$ by $E(\xi_1, x_2, \xi_2) = H(A(\xi_1, x_2, \xi_2), \xi_1)$. One can easily show that:

$$F(\xi_1, x_2, \xi_2) = \frac{\partial_{\xi_1} H(A, \xi_1)}{\partial_A H(A, \xi_1)}\bigg|_{A=A(\xi_1,x_2,\xi_2)},$$

and the following variational characterization of Λ:

Let us fix the energy to E and let $\Gamma(\xi_1, E)$ be the energy shell $\{E(\xi_1, x_2, \xi_2) = E\}$. Then

$$\Lambda = \bigcup_{E,i} \big\{(\Gamma\big(\xi_1^i, E\big), \xi_1^i\big\}, \quad \text{where } \xi_1^i \text{ is such that the area inside } \Gamma\big(\xi_1^i, E\big) \text{ is extremal.}$$

4. Dispersion of Poincaré waves and proof of the theorem

The proof of the theorem involves, for the Rossby modes ($j=0$), the standard result of propagation of the frequency set. If the initial frequency set is such that part of the Rossby mode is trapped, in particular if it intersects $\{b'(x_2)=0\}$, this concludes the proof.

If not we have to prove some dispersion for the Poincaré modes ($j=\pm$) for times of the order $\frac{1}{\epsilon}$, for which the theorem of propagation of the frequency set is not enough. But, the system being integrable, we can perform an expansion on eigenfunctions of the Hamiltonians $T_\pm$ and a decomposition of $U_{|t=0}$ on a compact set of coherent states (thanks to the condition on the microlocalization of the initial datum) [5]. Let us give a partial idea of the proof.

Since $\tau_\pm = \pm\sqrt{\xi_2^2+b^2(x_2)+\xi_1^2}+\mathrm{O}(\epsilon)$ and we are microlocalized far away from $\xi_2^2+b^2(x_2)+\xi_1^2=0$, we can find pseudo-differential operators $H_{2\pm}$ of principal symbols $\xi_2^2+b^2(x_2)$ such that $T_\pm=\pm\sqrt{H_{2\pm}+\xi_1^2}$. Far away from the separatrices of $H_{2\pm}$ (see [1] for the full proof using the hypothesis of nondegeneracy of the critical points of b^2), the Bohr–Sommerfeld quantization condition (with subsymbol) gives that the eigenvalues of $H_{2\pm}$ are of the form:

$$\lambda_\pm^k = \lambda_\pm\left(\left(k+\frac{1}{2}\right)\epsilon\right)+\epsilon\mu_\pm^k(\xi_1)+\mathrm{O}\big(\epsilon^2\big),$$

where $\lambda_\pm$ is the energy $\xi_2^2+b^2(x_2)$ defined on action variable, and $\epsilon\mu_\pm^k(\xi_1)\in C^\infty$ is the correction due to the subsymbol. Propagating at time $t=s/\epsilon$ a function, product of a coherent state at (q,p) (in x_1) and an eigenfunction of $T_\pm$ (in x_2), gives rise to expressions of the type:

$$\int \exp i\frac{\epsilon(x_1-q)\xi_1 \pm (\lambda_\pm^k+\xi_1^2)^{\frac{1}{2}}s + i\epsilon(\xi_1-p)^2}{\epsilon}\,\mathrm{d}\xi_1.$$

The stationary phase lemma then gives that this integral is $\mathrm{O}(\epsilon^\infty)$ except if there exists a stationary point, given by the conditions:

$$\xi_1=p \quad\text{and}\quad \epsilon(x_1-q)\pm\frac{2\xi_1+\epsilon\partial_{\xi_1}\mu_\pm^k}{2\sqrt{\lambda_\pm^k+\xi_1^2}}s=0.$$

The second condition gives: $2\sqrt{\lambda_\pm^k+\xi_1^2}(x_1-q)=\mp(\frac{2p}{\epsilon}+\partial_{\xi_1}\mu_\pm^k)s$. Therefore, since $p\neq 0$ and the λ^k 's are bounded by the above condition (3) on $\mathcal{C}$, there is no critical point for x_1 in a compact set.

Appendix A. A microlocal analysis lemma

In this appendix we give the crucial lemma for the proof of Proposition 2.1. It tells us that the principal symbols of $T_\pm$, T_0 can be computed by solving the symbolic equation associated to

$$\tau^3u_2+\tau\big(\epsilon^2\partial_{x_2}^2+\epsilon^2\partial_{x_1}^2-b^2(x_2)\big)u_2-i\epsilon^2b'(x_2)\partial_{x_1}u_2=0$$

obtained by (exact) algebraic computations from (1), that is

$$\tau^3-\tau\big(\xi_1^2+\xi_2^2+b^2\big)+\epsilon b'\xi_1=0. \tag{A.1}$$

Lemma A.1. *Let $h=h(x,\xi,\sigma)$ be a smooth function such that $\partial_\sigma h|_{h=0}\neq 0$, and let $\tau=\tau(x,\xi)$ be any continuous root of $h(x,\xi,\tau(x,\xi))=0$.*

Then there exists a pseudo-differential operator $T=T(x,-i\epsilon\partial_x)$ with principal symbol $\tau(x,\xi)$ such that:

$$T\psi=\lambda\psi \implies H(\lambda)\psi=\mathrm{O}\big(\epsilon^\infty\big) \tag{A.2}$$

where $H(\lambda)$ is a pseudo-differential operator of full symbol $h(x,\xi,\lambda)$.

Sketch of the proof. The proof uses essentially pseudo-differential functional calculus [4].

For any (x,ξ) the principal symbol of $h(x,\xi,T)$ at (y,ξ') is $h(x,\xi,\tau(y,\xi'))$. We can write at first order: $H\psi := \int e^{i\frac{\xi(x-y)}{\epsilon}} h(x,\xi,T(y,-i\epsilon\partial_y))\psi(y)\frac{d\xi\,dy}{\epsilon} \int e^{i\frac{\xi(x-y)}{\epsilon}} e^{i\frac{\xi'(y-y')}{\epsilon}} h(x,\xi,\tau(y',\xi'))\psi(y')\frac{d\xi\,d\xi'\,dy}{\epsilon}$. The integral over y gives $\delta(\xi-\xi')$. Therefore: $H\psi(x) = \int e^{i\frac{\xi(x-y)}{\epsilon}} h(x,\xi,\tau(y,\xi))\psi(y)\frac{d\xi\,dy}{\epsilon}$. So the principal symbol of H is $h(x,\xi,\tau(x,\xi))$ which, by assumption, is 0.

For the ϵ^∞ result, it is enough to repeat the same argument taking into account lower order terms $\epsilon^k\tau_k$ for and adding to any symbol a term of the form $\sum_{k\geqslant 1}\epsilon^k P_k\tau$ where the P_ks are differential operators. We end up with an equation for $\tau_\epsilon \sim \tau + \sum \epsilon^k\tau_k$ of the form: $h(x,\xi,\tau_\epsilon) + \sum_{k\geqslant 1}\epsilon^k Q_k(\tau,\ldots,\partial_x^l\partial_\xi^m\tau_\epsilon) = 0$, that can be solved recursively under the condition $\partial_\tau h(x,\xi,\tau)|_{h(x,\xi,\tau)=0} \neq 0$.

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References

[1] C. Cheverry, I. Gallagher, T. Paul, L. Saint-Raymond, Trapping Rossby waves by wind forcing, in preparation.
[2] A. Dutrifoy, A.J. Majda, S. Schochet, A simple justification of the singular limit for equatorial shallow-water dynamics, Comm. Pure Appl. Math. LXI (2008) 0002–0012.
[3] I. Gallagher, L. Saint-Raymond, Mathematical study of the betaplane model: equatorial waves and convergence results, Mém. Soc. Math. France (N.S.) 107 (2006) v+116.
[4] A. Martinez, An Introduction to Semiclassical and Microlocal Analysis, Springer, 2002.
[5] T. Paul, Échelles de temps pour l'évolution quantique à petite constante de Planck, Séminaire X-EDP 2007–2008, École polytechnique.
[6] J. Pedlosky, Geophysical Fluid Dynamics, Springer, 1979.
[7] J. Pedlosky, Ocean Circulation Theory, Springer, 1996.