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Zero-noise solutions of linear transport equations without uniqueness: an example

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Abstract

We consider a classical one-dimensional example of linear transport equation without uniqueness of weak solutions. Under a suitable multiplicative noise perturbation, the equation is well posed. We identify the two solutions of the deterministic equation obtained in the zero-noise limit. In addition, we prove that the zero-viscosity solution exists and is different from them. ***To cite this article: S. Attanasio, F. Flandoli, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Solutions à bruit nul des équations linéaires de transport : un exemple. On considère un exemple unidimensionnel classique d'équation de transport linéaire sans unicité des solutions faibles. En présence d'une perturbation donnée par un bruit multiplicatif convenablement choisi, l'équation se révèle bien posée. On identifie les deux solutions de l'équation déterministe obtenues dans la limite ou le bruit s'annule. On prouve aussi que la solution de viscosité nulle existe et qu'elle est différente des deux autres. ***Pour citer cet article : S. Attanasio, F. Flandoli, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

1. Introduction

The transport equation in $[0,T]\times\mathbb{R}^d$

$$\partial_t u(t,x) + b(x)\cdot\nabla u(t,x) = 0, \quad u(0,x) = u_0(x) \tag{1}$$

is not necessarily well posed when the vector field $b:\mathbb{R}^d\to\mathbb{R}^d$ is not sufficiently regular. It is well posed in the class $L^\infty(0,T;L^\infty(\mathbb{R}^d))$ when $b\in W^{1,1}_{loc}(\mathbb{R}^d,\mathbb{R}^d)$ or even $b\in BV_{loc}(\mathbb{R}^d,\mathbb{R}^d)$, with a linear growth condition and $\operatorname{div} b\in L^\infty(\mathbb{R}^d)$, see R.J. Di Perna and P.L. Lions [3] and L. Ambrosio [1]. But if b is, for instance, only Hölder continuous, there are well known counterexamples to uniqueness, in the L^∞-class of solutions, see the next section.

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In such cases of non-uniqueness, it would be interesting to have selection criteria. One approach to selection is to approximate the equation by others which have unique solutions and analyze the limit of their solutions. Well posedness is restored if Eq. (1) is perturbed in two different ways: (i) the most classical one is by means of a viscous term $\varepsilon\triangle u(t,x)$ on the right-hand side of the equation, see for instance [9]; (ii) another one is by means of a stochastic multiplicative noise of the form $\varepsilon\nabla u(t,x)\circ\frac{\mathrm{d}W(t)}{\mathrm{d}t}$ where $(W(t))_{t\geqslant 0}$ is a d-dimensional Brownian motion, see [5]. Precise assumptions on b and the generalization to time dependent case can be found in these works (and others for case (i)). Denote by $u^{\varepsilon}_{visc}(t,x)$ and $u^{\varepsilon}_{stoch}(t,x)$ the unique L^∞-solutions provided by methods (i) and (ii) respectively. One would like to see whether u^{ε}_{visc} and u^{ε}_{stoch} converge as $\varepsilon\to 0$, which limit points are identified, whether the limit points of u^{ε}_{visc} and u^{ε}_{stoch} are the same or not.

In general this is a very difficult problem. In the next sections we give a complete answer in a particular example. Generalizations and other issues will be reported elsewhere, see also the remarks at the end of the paper. The result in the example is that the probability law of u^{ε}_{stoch} converges weakly to the convex combination $\frac{1}{2}\delta_{u_1}+\frac{1}{2}\delta_{u_2}$ where u_1 and u_2 are two special solutions of Eq. (1), while u^{ε}_{visc} converges to a single solution $u_\triangle$, average of the two previous ones. The zero-viscosity solution is thus perhaps of less 'physical' content than the two solutions identified by the zero-noise procedure, but the meaning of this result must be understood better. Let us mention that for non-well posed ODEs it has already been observed that deterministic regularization of the driving vector field and stochastic perturbation may lead to different objects in the limit, see [2,1,4]. The present one seems to be the first result of this nature for PDEs.

2. The example and some preliminary facts

Consider the function $b\in W^{1,1}_{loc}(\mathbb{R})$ and the discontinuous initial condition u_0 defined as

$$b(x)=\operatorname{sgn}(x)\big(|x|\wedge R\big)^{\gamma},\quad u_0=\mathbb{1}_{[0,\infty)} \tag{2}$$

where $R>0$ and $\gamma\in(0,1)$ are two given numbers. The boundedness of b is imposed only to use more classical bibliographical references. Consider the equation on a time interval $[0,T]$. To simplify, assume that $(T(1-\gamma))^{\frac{1}{1-\gamma}}\leqslant R$, so that the two extreme solutions $\psi_1\geqslant 0$ and $\psi_2\leqslant 0$ of the deterministic equation driven by b with zero initial condition are equal, on $[0,T]$, to $\pm(t(1-\gamma))^{\frac{1}{1-\gamma}}$.

Consider the transport equation (1) above in the case $d=1$, with a generic $b\in W^{1,1}_{loc}(\mathbb{R})$, like the one given by (2). Denote by $\mathcal{D}([0,T)\times\mathbb{R})$ the space of all $\varphi\in C^\infty_c([0,T]\times\mathbb{R})$ with support in $[0,T)\times\mathbb{R}$. We look for solutions $u\in L^\infty([0,T]\times\mathbb{R})$ such that for every $\varphi\in\mathcal{D}([0,T)\times\mathbb{R})$ we have

$$\int_0^T\!\!\int_{\mathbb{R}} u\,\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\!\!\int_{\mathbb{R}} u\,\partial_x(b\varphi)\,\mathrm{d}x\,\mathrm{d}t+\int_{\mathbb{R}}\varphi(0,x)u_0(x)\,\mathrm{d}x=0 \tag{3}$$

with $u_0\in L^\infty(\mathbb{R})$ (see [3]). This definition is meaningful since $b\in W^{1,1}_{loc}(\mathbb{R})$. Depending on the continuity of u_0 at $x=0$, this problem may have more than one solution (the three solutions below are examples).

Given a one-dimensional Brownian motion $(\Omega,\mathcal{F},P,\mathcal{F}_t,W)$, let us introduce the stochastic transport equation

$$\frac{\partial u^\varepsilon}{\partial t}+b(x)\frac{\partial u^\varepsilon}{\partial x}=\varepsilon\frac{\partial u^\varepsilon}{\partial x}\circ\frac{\mathrm{d}W}{\mathrm{d}t},\quad u^\varepsilon(x,0)=u_0(x) \tag{4}$$

where $u^\varepsilon=u^\varepsilon(t,x,\omega)$, $t\geqslant 0$, $x\in\mathbb{R}$, $\omega\in\Omega$. The stochastic multiplicative term will be understood in the Stratonovich sense. The concept of solution is given in Theorem 2.2 below. Consider also the parabolic equation

$$\partial_t v^\varepsilon(t,x)+b(x)\partial_x v^\varepsilon(t,x)=\big(\varepsilon^2/2\big)\triangle v^\varepsilon,\quad v^\varepsilon(0,x)=u_0(x). \tag{5}$$

Both Eqs. (4) and (5) are related to the stochastic ordinary differential equation

$$\mathrm{d}X^{x,\varepsilon}_t=b\big(X^{x,\varepsilon}_t\big)\,\mathrm{d}t+\varepsilon\,\mathrm{d}W_t,\quad X^{x,\varepsilon}_0=x. \tag{6}$$

It is well known that, for every $x\in\mathbb{R}$ and $\varepsilon>0$, Eq. (6) has a unique strong continuous adapted solution, see [10]. We use the following improvement of this classical result, taken from [5]; see also [6]. We add property (iii) of monotonicity which is an easy consequence of the diffeomorphism property, or of the comparison principle for stochastic equations (see [10], Chapter 9).

Theorem 2.1. *Eq.* (6) *generates a stochastic flow of diffeomorphisms. In particular there is a real valued map* $(t,x,\omega)\to\phi_t^\varepsilon(x)(\omega)$ *defined for* $t\geqslant 0$, $x\in\mathbb{R}$, $\omega\in\Omega$, *such that*:

(i) *for every* $x\in\mathbb{R}$, *the process* $X_t^{x,\varepsilon}:=\phi_t^\varepsilon(x)$ *is a continuous adapted solution of Eq.* (6);
(ii) P*-a.s.,* $\phi_t^\varepsilon(x)$ *is a diffeomorphism for every* $t\geqslant 0$ *and the functions* $\phi_t^\varepsilon(x)$, $(\phi_t^\varepsilon)^{-1}(x)$, $D\phi_t^\varepsilon(x)$, $D(\phi_t^\varepsilon)^{-1}(x)$, *are continuous in* (x,t);
(iii) P*-a.s.,* $x\mapsto\phi_t^\varepsilon(x)$ *is strictly increasing, for every* $t\geqslant 0$.

We base our main result on the following theorem from [5]; we remark that it holds true also in dimension d for any Hölder continuous and bounded b with $\operatorname{div} b\in L^p_{loc}(\mathbb{R}^d)$ for some $p>d$:

Theorem 2.2. *Given* $u_0\in L^\infty(\mathbb{R})$, *the process* $u^\varepsilon(t,x):=u_0((\phi_t^\varepsilon)^{-1}(x))$ *is a solution of* (4), *in the following sense*:

1. $u^\varepsilon(t,x,\omega)$, *with* $t\in[0,T]$, $x\in\mathbb{R}$, $\omega\in\Omega$, *is measurable and* $\sup_{t\in[0,T],x\in\mathbb{R},\omega\in\Omega}|u^\varepsilon(t,x,\omega)|<\infty$;
2. *for every* $\theta\in C_0^0(\mathbb{R})$, *the process* $t\to\int_{\mathbb{R}}u^\varepsilon(t,x)\theta(x)\,\mathrm{d}x$ *is continuous and adapted to* $\mathcal{F}_t$;
3. *for every* $\theta\in C_0^\infty(\mathbb{R})$, *the process* $t\to\int_{\mathbb{R}}u^\varepsilon(t,x)\theta(x)\,\mathrm{d}x$ *is a continuous semimartingale with respect to* $\mathcal{F}_t$ *and for every* $t\in[0,T]$ *we have*:

$$\int_{\mathbb{R}}u^\varepsilon(t)\theta\,\mathrm{d}x=\int_0^t\mathrm{d}s\int_{\mathbb{R}}u^\varepsilon[b\theta'+b'\theta]\,\mathrm{d}x+\int_{\mathbb{R}}u_0\theta\,\mathrm{d}x+\int_0^t\varepsilon\left(\int_{\mathbb{R}}u^\varepsilon\theta'\,\mathrm{d}x\right)\circ\mathrm{d}W_s. \tag{7}$$

Let us start with the following classical result on Eq. (5):

Theorem 2.3. *Given* $u_0\in L^\infty(\mathbb{R})$, *the flow* ϕ_t^ε *and the solution* u^ε *of the previous theorems, the function*

$$v^\varepsilon(t,x)=E\big[u_0\big(\big(\phi_t^\varepsilon\big)^{-1}(x)\big)\big]=E\big[u^\varepsilon(t,x)\big] \tag{8}$$

is a weak solution of Eq. (5).

For the limited purposes of this note, by weak solution we simply understand a function of class L^∞ which satisfies a weak formulation of Eq. (5) similar to (3) above. The second identity in (8) comes from Theorem 2.2. The proof that $v^\varepsilon(t,x)=E[u_0((\phi_t^\varepsilon)^{-1}(x))]$ is a weak solution of Eq. (5) is a classical fact. Alternatively, here we may check directly that $v^\varepsilon(t,x)=E[u^\varepsilon(t,x)]$ is a weak solution of Eq. (5), by taking expectation in Eq. (7) and using the fact that for two continuous semimartingales X and Y we have $\int_0^t Y_s\circ\mathrm{d}X_s=\int_0^t Y_s\,\mathrm{d}X_s+\frac{1}{2}\langle Y,X\rangle_t$. The details can be left to the reader.

Since, from the probabilistic representation, $|v^\varepsilon(t,x)|\leqslant\sup_{\mathbb{R}}|u_0|$ for every $\varepsilon>0$, every L^1_{loc}-limit point of $(v^\varepsilon)_{\varepsilon>0}$ is an L^∞-solution of (1). The following theorem shows that, in our example, v^ε has a single limit point:

Theorem 2.4. *If* b *and* u_0 *are given by* (2), *then* $v^\varepsilon(t,x)$ *converges a.s. to the following* L^∞*-solution of* (1): $u_\triangle(t,x)=\mathbb{1}_{\{x\geqslant\psi_1(t)\}}+(1/2)\mathbb{1}_{\{\psi_2(t)<x<\psi_1(t)\}}$.

Proof. From (8), (2) and (iii) of Theorem 2.1, we have $v^\varepsilon(t,x)=E[u_0((\phi_t^\varepsilon)^{-1}(x))]=P\{(\phi_t^\varepsilon)^{-1}(x)\geqslant 0\}=P\{\phi_t^\varepsilon(0)\leqslant x\}$. In [2] it is proved that $\phi_t^\varepsilon(0)$ converges in law to $\frac{1}{2}\delta_{\psi_1}+\frac{1}{2}\delta_{\psi_2}$, for $t\in[0,T]$; see also [7] and [8]. Then $P\{\phi_t^\varepsilon(0)\leqslant x\}\to\frac{1}{2}$ if $x\in(\psi_2(t),\psi_1(t))$, $P\{\phi_t^\varepsilon(0)\leqslant x\}\to 1$ if $x>\psi_1(t)$ and $P\{\phi_t^\varepsilon(0)\leqslant x\}\to 0$ if $x<\psi_2(t)$. The proof is complete. □

Let us now investigate the limit as $\varepsilon\to 0$, in law, of the solutions u^ε of Eq. (4). It is convenient to consider the mapping $\omega\mapsto u^\varepsilon(\omega)$ from $(\Omega,\mathcal{F}_T,P)$ to $L^1_{loc}([0,T]\times\mathbb{R})$. From Theorem 2.1, given $\varepsilon>0$, the mapping $\Phi^\varepsilon:(\Omega,\mathcal{F}_T,P)\to C([0,T]\times\mathbb{R})$ defined as $(\Phi^\varepsilon(\omega))(t,x)=((\phi_t^\varepsilon)^{-1}(x))(\omega)$ is $\mathcal{F}_T$ measurable. Since $u_0\in L^\infty$, P-a.s. we have $u^\varepsilon=u_0\circ\Phi^\varepsilon\in L^1_{loc}([0,T]\times\mathbb{R})$ and u^ε can be seen as a measurable map from $(\Omega,\mathcal{F}_T,P)$ to $L^1_{loc}([0,T]\times\mathbb{R})$. Let P^ε be its law on Borel sets of $L^1_{loc}([0,T]\times\mathbb{R})$.

Theorem 2.5. *If b and u_0 are given by* (2), *then P^ε converges weakly on $L^1_{loc}([0,T]\times\mathbb{R})$ to $\frac{1}{2}\delta_{u_1}+\frac{1}{2}\delta_{u_2}$ as $\varepsilon\to 0$, where*

$$u_1(t,x)=\mathbb{1}_{\{x\geqslant\psi_1(t)\}},\qquad u_2(t,x)=\mathbb{1}_{\{x>\psi_2(t)\}}.$$

Proof. Step 1. By definition of u^ε and u_i, for $i=1,2$ we have the identity $\int_{\mathbb{R}}|u^\varepsilon(t,x,\omega)-u_i(t,x)|\,\mathrm{d}x=|\phi_t^\varepsilon(0)(\omega)-\psi_i(t)|$, hence, denoting the $L^1([0,T]\times\mathbb{R})$-norm by $\|.\|_1$, $\|u^\varepsilon(\omega)-u_i\|_1=\int_0^T|\phi_t^\varepsilon(0)(\omega)-\psi_i(t)|\,\mathrm{d}t$.
Step 2. Given a probability space (Ω,F,P), a metric space (S,d), two points $x_1,x_2\in X$ at positive distance $r=d(x_1,x_2)$, and a family of r.v. $(X_\varepsilon)_{\varepsilon>0}$ on (Ω,F,P) with values in S, with laws $(\mu_\varepsilon)_{n\geqslant 1}$, the property of weak convergence $\mu_\varepsilon\to\frac{1}{2}\delta_{x_1}+\frac{1}{2}\delta_{x_2}$ is equivalent to the property that for all $\delta\in(0,\frac{r}{2})$, $\lim_{\varepsilon\to 0}P(d(X_\varepsilon,x_i)<\delta)=\frac{1}{2}$ for both $i=1,2$. Standing this general fact, from [2] we know that the law of $\phi_\cdot^\varepsilon(0)$ converges weakly to $\frac{1}{2}\delta_{\psi_1}+\frac{1}{2}\delta_{\psi_2}$ on the space $C([0,T])$, hence on the space $L^1([0,T])$. Hence, if r is the $L^1([0,T])$-distance between ψ_1 and ψ_2 and $\delta\in(0,\frac{r}{2})$, using step 1 we have $\lim_{\varepsilon\to 0}P(\|u^\varepsilon(\omega)-u_i\|_1<\delta)=\lim_{\varepsilon\to 0}P(\int_0^T|\phi_t^\varepsilon(0)-\psi_i(t)|\,\mathrm{d}t<\delta)=\frac{1}{2}$ for both $i=1,2$. This implies the claim of the theorem, again by the general fact above. The proof is complete. □

We have seen that: (i) the limit law of the stochastic approximation is concentrated over two solutions u_1 and u_2 of Eq. (1); (ii) the viscous approximation converges to the average $u_\triangle$ of u_1 and u_2. Being zero-noise limit, u_1 and u_2 could have a deeper 'physical' meaning than $u_\triangle$.

Remark 1. In spite of the property $u_0(x)\in[0,1]$, the weak L^∞ solutions of Eq. (1) do not necessarily take values in $[0,1]$: for instance the function $u(t,x)=\mathbb{1}_{\{x>\psi_1(t)\}}+a\cdot\mathbb{1}_{\{\psi_2(t)<x<\psi_1(t)\}}$ is a solution for every $a\in\mathbb{R}$. The fact that $u_\triangle$ is a *convex* linear combination of u_1 and u_2 is a special fact, not a property of all possible solutions. The previous example shows that u_1 and u_2 are not extremal of the set of all solutions.

Remark 2. From the large deviation result of [7] we may easily deduce some large deviation statements for u^ε. We have not optimized the results, so we give only one example. Given the distance $d(u,v)=\int_{-\infty}^{+\infty}\frac{(|u(x)-v(x)|\wedge 1)}{1+x^2}\,\mathrm{d}x$ on $L^1_{loc}(\mathbb{R})$, let $u_r=\mathbb{1}_{\{x\in[r,+\infty)\}}$ and let $A=\{v\in L^1_{loc}(\mathbb{R})\colon d(u_r,v)<\delta\}$. It is easy to prove that $\lim_{\varepsilon\to 0}\varepsilon^{\beta_i}\ln P\{u^\varepsilon(t,\cdot)\in A\}=-\inf_{(\alpha^-(r,\delta),\alpha^+(r,\delta))}k_i$ where $\beta_i=2$ for $\alpha^-(r,\delta)>\psi_1(t)$, $\beta_i=2\frac{1-\gamma}{1+\gamma}$ for $\alpha^+(r,\delta)<\psi_1(t)$, $\alpha^\pm(r,\delta)=\tan(\arctan(|r|)\pm\delta)$ and k_1, k_2, are the functions explicitly given in [7]. The expression for α^+ and α^- depends on the distance; this is one of the simplest examples.

Remark 3. The result of this paper extends to all b satisfying the hypotheses of [5] and [2], so that b has only one zero, say at $x=0$, and $\phi_\cdot^\varepsilon(0)$ converges in law to a non-trivial convex combination $\alpha\delta_{\psi_1}+(1-\alpha)\delta_{\psi_2}$, $\alpha\in(0,1)$. When $\phi_\cdot^\varepsilon(0)$ simply converges in law to some δ_{ψ_1}, u^ε also converges in law to some δ_u, and u is also the zero-viscosity solution. The case of more than one zero of b is different and may be more complex.

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