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Partial Differential Equations

Regularity of solutions for the Boltzmann equation without angular cutoff

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Abstract

We prove that classical solution of the spatially inhomogeneous and angular non-cutoff Boltzmann equation is C^∞ with respect to all variables, locally in the space and time variables. The proof relies on a generalized uncertainty principle, some improved upper bound and coercivity estimates on the nonlinear collision operator, and some subtle analysis on the commutators between the collision operators and some appropriately chosen pseudo-differential operators. ***To cite this article: R. Alexandre et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Régularité de solutions pour l'équation de Boltzmann sans angulaire cutoff. Nous considérons l'équation de Boltzmann inhomogène sans hypothèse de troncature angulaire. Nous montrons que toute solution classique est C^∞ par rapport à toutes les variables, localement en temps et en espace. La preuve s'appuie sur un principe d'incertitude généralisé, des bornes fonctionnelles précisées sur l'opérateur de collision, une estimation de coercivité, ainsi qu'une analyse de commutateurs avec cet opérateur, avec un choix approprié d'opérateurs pseudo-différentiels. ***Pour citer cet article : R. Alexandre et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

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On considère l'équation de Boltzmann inhomogène et non linéaire (1) (les numéros renvoient à la version longue anglaise). L'opérateur de collision est donné par (2).

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Nous supposons que le noyau de collision B est donné par les formules (4), (5), et donc en particulier, nous ne faisons pas l'hypothèse de troncature angulaire de Grad. Le résultat de cette Note concerne la régularisation d'une solution classique, fait bien connu maintenant dans le cas homogène, et initié par les travaux de L. Desvillettes [11–13]. Plus précisément,

Théorème 0.1. *On suppose que* $0 < s < 1$, $\gamma \in \mathbb{R}$, $0 < T \leqslant \infty$ *et soit* $\Omega \subset \mathbb{R}^3_x$ *un ouvert. Soit* $f \in \mathcal{H}^5_l(]0, T[\times \Omega \times \mathbb{R}^3_v)$ *une solution positive de l'équation de Boltzmann non linéaire et inhomogène* (1) *dans le domaine* $]0, T[\times \Omega \times \mathbb{R}^3_v$, *pour tout* $l \in \mathbb{N}$. *On suppose de plus*

$$\|f(t, x, \cdot)\|_{L^1(\mathbb{R}^3_v)} > 0,$$

pour tout $(t, x) \in]0, T[\times \Omega$. *Alors, il s'ensuit que*

$$f \in C^\infty\big(]0, T[\times \Omega; \mathcal{S}\big(\mathbb{R}^3_v\big)\big).$$

Tous les détails de cette Note sont explicités dans la prépublication [5].

1. Introduction and main result

Consider the nonlinear spatially inhomogeneous and angular non-cutoff Boltzmann equation

$$f_t + v \cdot \nabla_x f = Q(f, f). \tag{1}$$

Here, $f = f(t, x, v)$ is the density distribution of particles with position $x \in \mathbb{R}^3$ and velocity $v \in \mathbb{R}^3$ at time $t \in]0, T[$ for some fixed $0 < T \leqslant +\infty$. The right-hand side of (1) is given by the Boltzmann bilinear collision operator, acting only on the variable v,

$$Q(g, f) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v - v_*, \sigma)\big\{g(v'_*)f(v') - g(v_*)f(v)\big\}\,\mathrm{d}\sigma\,\mathrm{d}v_*, \tag{2}$$

for some suitable functions f and g. Here B is the non-cutoff cross-section, and for given $(v, v_*) \in \mathbb{R}^3$, $\sigma \in \mathbb{S}^2$, the relations between the post- and pre-collisional velocities are given by

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\sigma, \qquad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2}\sigma, \tag{3}$$

which come from the conservation of momentum and energy through particle collisions, see [8,16] and references therein.

For the spatially homogeneous case, the full regularization of weak solutions to C^∞ in all (time and velocity) variables for positive time is now well understood through the pioneering works of L. Desvillettes, and works developed later; see [6,11–14,16] and references therein.

In this Note, we will prove the C^∞ regularity of solutions with respect to all (time, space and velocity) variables by requiring some regularity on the solutions to start with. This is similar to the situation in [10] on the Landau equation whose collision operator, however, is a differential operator, see also [9] and references therein for coupled type equations. Let us mention that in the cutoff case, propagation of regularity is studied in [7].

The assumption imposed on the cross-section in this Note is as follows: B is a non-negative function of the form

$$B\big(|v - v_*|, \cos\theta\big) = \Phi\big(|v - v_*|\big)b(\cos\theta), \quad \cos\theta = \left\langle \frac{v - v_*}{|v - v_*|}, \sigma \right\rangle, \quad 0 \leqslant \theta \leqslant \frac{\pi}{2}, \tag{4}$$

while the kinetic factor Φ and the angular factor b satisfy

$$\Phi\big(|v - v_*|\big) = \big(1 + |v - v_*|^2\big)^{\frac{\gamma}{2}}, \qquad b(\cos\theta) \approx K\theta^{-2-2s} \quad \text{when } \theta \to 0+, \tag{5}$$

where $K > 0$ is a constant. Φ is assumed to be regular near 0 in order to be able to use functional estimates on the collision operator, see [2,5]. Furthermore, γ is allowed to be any real number.

Throughout the Note, we shall use the following standard weighted (with respect to the velocity variable $v \in \mathbb{R}^3_x$) Sobolev spaces. For $m \in \mathbb{R}$, $l \in \mathbb{R}$, set $W_l = \langle v\rangle^{\frac{l}{2}} = (1+|v|^2)^{\frac{l}{2}}$ and

$$H^m_l(\mathbb{R}^7) = \{f \in \mathcal{S}'(\mathbb{R}^7);\ W_l f \in H^m(\mathbb{R}^7_{t,x,v})\}.$$

We also use the function spaces $H^m_l(\mathbb{R}^6_{x,v})$ and $H^m_l(\mathbb{R}^3_v)$ when the variables are specified, where the weight is always with respect to $v \in \mathbb{R}^3$. Since the regularity proved later on the solution is local in space and time, we define a local weighted Sobolev space. That is, for $0 < T \leqslant +\infty$ and any open domain $\Omega \subset \mathbb{R}^3_x$, set

$$\mathcal{H}^m_l(]0,T[\times\Omega\times\mathbb{R}^3_v) = \{f \in \mathcal{D}'(]0,T[\times\Omega\times\mathbb{R}^3_v);\ \varphi(t)\psi(x)f \in H^m_l(\mathbb{R}^7),\ \forall\varphi \in C^\infty_0(]0,T[),\ \psi \in C^\infty_0(\Omega)\}.$$

The main aim of this Note is to show that the smoothing effect indeed holds for the angular non-cutoff and spatially inhomogeneous Boltzmann equation. More precisely, we have

Theorem 1.1. *Assume that $0 < s < 1$, $\gamma \in \mathbb{R}$, $0 < T \leqslant +\infty$, $\Omega \subset \mathbb{R}^3_x$ is an open domain. Let f be a non-negative function belonging to $\mathcal{H}^5_l(]0,T[\times\Omega\times\mathbb{R}^3_v)$ for all $l \in \mathbb{N}$ and solving* (1) *in the domain $]0,T[\times\Omega\times\mathbb{R}^3_v$ in the classical sense. Assume furthermore that*

$$\|f(t,x,\cdot)\|_{L^1(\mathbb{R}^3_v)} > 0, \tag{6}$$

for all $(t,x) \in]0,T[\times\Omega$. Then, it follows that

$$f \in C^\infty(]0,T[\times\Omega;\mathcal{S}(\mathbb{R}^3_v)).$$

The full proof of this theorem, among other results, in particular an existence result with the required assumptions, is detailed in [5]. Note in particular the two existence results [1,15] about local L^∞ and Gevrey solutions, respectively.

2. Ingredients of the proof

Among the technical tools used in the analysis, let us begin by stating the following upper bound estimate on the nonlinear collision operator (see also [2,14]):

Theorem 2.1. *Let $0 < s < 1$ and $\gamma \in \mathbb{R}$. Then for any $m \in \mathbb{R}$ and any $\alpha \in \mathbb{R}$, one has*

$$\|Q(f,g)\|_{H^m_\alpha(\mathbb{R}^3_v)} \leqslant C\|f\|_{L^1_{\alpha^+ + (\gamma+2s)^+}(\mathbb{R}^3_v)}\|g\|_{H^{m+2s}_{(\alpha+\gamma+2s)^+}(\mathbb{R}^3_v)}. \tag{7}$$

The second result is about a coercivity estimate improving that from [3]:

Theorem 2.2. *Assume that $\gamma \in \mathbb{R}$, $0 < s < 1$. Let $g \geqslant 0, \not\equiv 0$, $g \in L^1_{\max\{\gamma^+,2-\gamma^+\}} \cap L\log L(\mathbb{R}^3_v)$. Then there exists a constant $C_g > 0$ depending only on $B(v-v_*,\sigma)$, $\|g\|_{L^1_1}$ and $\|g\|_{L\log L}$ such that for any smooth function $f \in H^1_{\gamma/2}(\mathbb{R}^3_v) \cap L^2_{\gamma^+/2}(\mathbb{R}^3_v)$, we have*

$$-(Q(g,f),f)_{L^2(\mathbb{R}^3_v)} \geqslant C_g\|W_{\gamma/2}f\|^2_{H^s(\mathbb{R}^3_v)} - C\|g\|_{L^1_{\max\{\gamma^+,2-\gamma^+\}}(\mathbb{R}^3_v)}\|f\|^2_{L^2_{\gamma^+/2}(\mathbb{R}^3_v)}. \tag{8}$$

We also need the following commutator estimates between some appropriately chosen pseudo-differential operators and the Boltzmann collision operator:

Proposition 2.1. *Let $\lambda \in \mathbb{R}$ and $M(\xi)$ be a positive symbol of a pseudo-differential operator in $S^\lambda_{1,0}$ in the form of $M(\xi) = \tilde{M}(|\xi|^2)$. Assume that for any $c > 0$, there exists $C > 0$ such that for any $s, \tau > 0$*

$$c^{-1} \leqslant \left|\frac{s}{\tau}\right| \leqslant c \quad \text{implies} \quad C^{-1} \leqslant \left|\frac{\tilde{M}(s)}{\tilde{M}(\tau)}\right| \leqslant C,$$

and that $M(\xi)$ satisfies

$$\left|\partial_\xi^\alpha M(\xi)\right| \leqslant C_\alpha M(\xi)\langle\xi\rangle^{-|\alpha|},$$

for any $\alpha \in \mathbb{N}^3$. Then, for any $N > 0$, if $0 < s < 1/2$, we have

$$\left|\left(M(D_v)Q(f,g) - Q\big(f, M(D_v)g\big), h\right)_{L^2(\mathbb{R}_v^3)}\right| \\ \leqslant C_N \|f\|_{L^1_{\gamma^+}(\mathbb{R}_v^3)}\big(\|Mg\|_{L^2_{\gamma^+}(\mathbb{R}_v^3)} + \|g\|_{H^{\lambda-N}_{\gamma^+}(\mathbb{R}_v^3)}\big)\|h\|_{L^2(\mathbb{R}_v^3)}.$$

Furthermore, if $1/2 < s < 1$, then for any $\varepsilon > 0$, there exists a constant $C > 0$ such that

$$\left|\left(M(D_v)Q(f,g) - Q\big(f, M(D_v)g\big), h\right)_{L^2(\mathbb{R}_v^3)}\right| \\ \leqslant C_{N,\varepsilon} \|f\|_{L^1_{(2s+\gamma-1)^+}(\mathbb{R}_v^3)}\big(\|Mg\|_{H^{2s-1+\varepsilon}_{(2s+\gamma-1)^+}(\mathbb{R}_v^3)} + \|g\|_{H^{\lambda-N}_{\gamma^+}(\mathbb{R}_v^3)}\big)\|h\|_{L^2(\mathbb{R}_v^3)}.$$

When $s = 1/2$ we have the same estimate with $(2s+\gamma-1)$ replaced by $(\gamma+\kappa)$ for any small $\kappa > 0$.

Finally, a key ingredient in the analysis is a hypo-elliptic result on the transport equation

$$f_t + v \cdot \nabla_x f = g \in \mathcal{D}'\big(\mathbb{R}^7\big), \tag{9}$$

where $(t,x,v) \in \mathbb{R}^7$. Actually, in [4], by using a generalized uncertainty principle, we proved the following hypo-elliptic estimates:

Theorem 2.3. *Assume that $g \in H^{-s'}(\mathbb{R}^7)$ for some $0 \leqslant s' < 1$. Let $f \in L^2(\mathbb{R}^7)$ be a weak solution of the transport equation (9) such that $\Lambda_v^s f \in L^2(\mathbb{R}^7)$ for some $0 < s \leqslant 1$. Then it follows that*

$$\Lambda_x^{\frac{s(1-s')}{(s+1)}} f \in L^2_{-\frac{ss'}{s+1}}\big(\mathbb{R}^7\big), \qquad \Lambda_t^{\frac{s(1-s')}{(s+1)}} f \in L^2_{-\frac{s}{s+1}}\big(\mathbb{R}^7\big),$$

where $\Lambda_\bullet = (1+|D_\bullet|^2)^{1/2}$.

3. Regularity of solutions

Let $f \in \mathcal{H}_l^5(]T_1,T_2[\times\Omega\times\mathbb{R}_v^3))$, for all $l \in \mathbb{N}$, be a non-negative solution to the Boltzmann equation (1). A function ϕ is a cutoff function if $\phi \in C_0^\infty$ and $0 \leqslant \phi \leqslant 1$. Notation $\phi_1 \Subset \phi_2$ stands for two cutoff functions such that $\phi_2 = 1$ on the support of ϕ_1. Take cutoff functions $\varphi, \varphi_2, \varphi_3 \in C_0^\infty(]T_1,T_2[)$ and $\psi, \psi_2, \psi_3 \in C_0^\infty(\Omega)$ such that $\varphi \Subset \varphi_2 \Subset \varphi_3$ and $\psi \Subset \psi_2 \Subset \psi_3$. Set $f_1 = \varphi(t)\psi(x)f$, $f_2 = \varphi_2(t)\psi_2(x)f$ and $f_3 = \varphi_3(t)\psi_3(x)f$. For $\alpha \in \mathbb{N}^7$, $|\alpha| \leqslant 5$, set

$$g = \partial^\alpha\big(\varphi(t)\psi(x)f\big) = \partial^\alpha_{t,x,v}\big(\varphi(t)\psi(x)f\big) \in L^2_l\big(\mathbb{R}^7\big).$$

By the translation invariance of the collision operator Q with respect to the v variable, and the Leibniz formula with respect to the t, x variables, the following equation holds in the weak sense:

$$g_t + v \cdot \partial_x g = Q(f_2, g) + G, \quad (t,x,v) \in \mathbb{R}^7, \tag{10}$$

where

$$G = \sum_{\alpha_1+\alpha_2=\alpha,\ 1\leqslant|\alpha_1|} C_{\alpha_2}^{\alpha_1} Q\big(\partial^{\alpha_1} f_2, \partial^{\alpha_2} f_1\big) + \partial^\alpha\big(\varphi_t\psi(x)f + v\cdot\psi_x(x)\varphi(t)f\big) + \big[\partial^\alpha, v\cdot\partial_x\big]\big(\varphi(t)\psi(x)f\big).$$

Since g is not smooth enough, we introduce a mollification of g with respect to the x, v variables so that we can use it as a test function in the weak formulation.

Let $S \in C_0^\infty(\mathbb{R})$ such that $0 \leqslant S \leqslant 1$, $S(\tau) = 1$ for $|\tau| \leqslant 1$ and $S(\tau) = 0$ if $|\tau| \geqslant 2$. It follows that

$$S_N(D_x)S_N(D_v) = S\big(2^{-2N}|D_x|^2\big)S\big(2^{-2N}|D_v|^2\big) : H_l^{-\infty}\big(\mathbb{R}^6\big) \to H_l^{+\infty}\big(\mathbb{R}^6\big),$$

is a regularization operator such that

$$\big\|\big(S_N(D_x)S_N(D_v)f\big)-f\big\|_{L^2_l(\mathbb{R}^6)}\to 0,\quad \text{as } N\to\infty.$$

By choosing another cutoff function $\psi \Subset \psi_1 \Subset \psi_2$ and setting $P_{N,l}=\psi_1(x)S_N(D_x)W_lS_N(D_v)$, we can take $\tilde g = P^\star_{N,l}(P_{N,l}g)\in C^1(\mathbb{R};H^{+\infty}(\mathbb{R}^6))$ as a test function for Eq. (10). Here $P^\star_{N,l}$ is the conjugate operator of $P_{N,l}$ in L^2 sense. Note that the regularity in the t variable follows from the equation. Then note that

$$\begin{aligned}-\big(Q(f_2,P_{N,l}g),P_{N,l}g\big)_{L^2(\mathbb{R}^7)} &= -\big(\big[S_N(D_v),v\big]\cdot\nabla_xS_N(D_x)g,\psi_1(x)W_lP_{N,l}g\big)_{L^2(\mathbb{R}^7)}\\ &\quad+\big(P_{N,l}Q(f_2,g)-Q(f_2,P_{N,l}g),P_{N,l}g\big)_{L^2(\mathbb{R}^7)}+(G,\tilde g)_{L^2(\mathbb{R}^7)}.\end{aligned}\tag{11}$$

The coercivity property of the operator $Q(f_2,\cdot)$ on the left-hand side and the upper bound of the terms on the right-hand side lead to

$$\Lambda^s_v f_1\in H^5_l\big(\mathbb{R}^7\big),\quad \forall l\in\mathbb{N},$$

which, together with the upper bound estimate (7), now gives

$$Q(f_2,g)+G\in H^{-s}_l\big(\mathbb{R}^7\big).$$

The next step is to apply the hypo-ellipticity result of Theorem 2.3 to Eq. (10). By some detailed estimation in particular on the commutators, we can deduce

$$\Lambda^{s_0}_{t,x}f_1\in H^5_l\big(\mathbb{R}^7\big),$$

for $s_0=\frac{s(1-s)}{(s+1)}$ and any $l\in\mathbb{N}$.

This partial smoothing effect in the t,x variable can be improved as follows:

Proposition 3.1. *Let* $0<\lambda<1$. *Suppose that for all* $l\in\mathbb{N}$ *and all cutoff functions* φ,ψ,

$$\Lambda^s_v\big(\varphi(t)\psi(x)f\big),\ \Lambda^\lambda_{t,x}\big(\varphi(t)\psi(x)f\big)\in H^5_l\big(\mathbb{R}^7\big).$$

Then, one has

$$\Lambda^s_v\Lambda^\lambda_{t,x}\big(\varphi(t)\psi(x)f\big)\in H^5_l\big(\mathbb{R}^7\big),$$

for any $l\in\mathbb{N}$ *and any cutoff functions* φ,ψ.

In view of this result, by iteration and differentiation (of fractional order) of Eq. (1), one can deduce a higher order regularity in the t,x variables,

$$\Lambda^{1+\varepsilon}_{t,x}\big(\varphi(t)\psi(x)f\big)\in H^5_l\big(\mathbb{R}^7\big),$$

for any $l\in\mathbb{N}$ and some $\varepsilon>0$.

Finally, again by using the coercivity property, we can prove

$$\Lambda^s_v\big(\varphi(t)\psi(x)f\big),\ \nabla_{t,x}\big(\varphi(t)\psi(x)f\big)\in H^5_l\big(\mathbb{R}^7\big)\quad\Rightarrow\quad \Lambda^{2s}_v\big(\varphi(t)\psi(x)f\big)\in H^5_l\big(\mathbb{R}^7\big),$$

and hence, again by induction,

$$\Lambda^{1+\varepsilon}_v\big(\varphi(t)\psi(x)f\big)\in H^5_l\big(\mathbb{R}^7\big),$$

for any $l\in\mathbb{N}$ and some $\varepsilon>0$. Thus $(\varphi(t)\psi(x)f)\in H^6_l(\mathbb{R}^7)$ for any $l\in\mathbb{N}$ and any cutoff function φ,ψ.

Now, the proof of Theorem 1.1 can be completed by an induction argument.

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