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C. R. Acad. Sci. Paris, Ser. I 347 (2009) 757–762
COMPTES RENDUS MATHEMATIQUE

Partial Differential Equations

A fixed point method for the $p(\cdot)$-Laplacian

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Received 26 March 2009; accepted 31 March 2009
Available online 14 May 2009
Presented by Jean-Pierre Kahane

Abstract

A topological method, based on the fundamental properties of the Leray–Schauder degree, is used in proving the existence of a week solution in $W_0^{1,p(\cdot)}(\Omega)$ to Dirichlet problem

$$-\operatorname{div}\big(|\nabla u|^{p(x)-2}\nabla u\big) = f(x,u), \quad x \in \Omega, \qquad (\mathcal{P})$$
$$u = 0, \quad x \in \partial\Omega.$$

This method is an adaptation of that used by Dinca et al. [G. Dinca, P. Jebelean, Une méthode de point fixe pour le p-laplacien, C. R. Acad. Sci. Paris, Ser. I 324 (1997) 165–168. [1], G. Dinca, P. Jebelean, J. Mawhin, Variational and topological methods for Dirichlet problems with p-Laplacian, Portugal. Math. 53 (3) (2001) 339–377. [2]] for Dirichlet problems with classical p-Laplacian ($p(x) \equiv p =$ const. > 1). ***To cite this article: G. Dinca, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Une méthode de point fixe pour le $p(\cdot)$-laplacien. On utilise une méthode topologique, basée sur les propriétés fondamentales du degré de Leray–Schauder, afin de démontrer l'existence d'une solution faible dans $W_0^{1,p(\cdot)}(\Omega)$ pour le problème de Dirichlet $(\mathcal{P})$. Cette méthode représente une adaptation de celle utilisée par Dinca et al. [G. Dinca, P. Jebelean, Une méthode de point fixe pour le p-laplacien, C. R. Acad. Sci. Paris, Ser. I 324 (1997) 165–168. [1], G. Dinca, P. Jebelean, J. Mawhin, Variational and topological methods for Dirichlet problems with p-Laplacian, Portugal. Math. 53 (3) (2001) 339–377. [2]] pour le p-laplacien classique ($p(x) \equiv p =$ cte. > 1). ***Pour citer cet article : G. Dinca, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Version française abrégée

En utilisant une méthode variationnelle, Fan et Zhang ont démontré dans [4] le théorème suivant :

Théorème 1. *Soit $\Omega \subset \mathbf{R}^N$, $N \geqslant 2$, un domaine borné et régulier et $p, q \in \mathcal{C}_+(\bar{\Omega}) = \{h \in \mathcal{C}(\bar{\Omega}) \mid h(x) > 1$ pour tout $x \in \bar{\Omega}\}$ satisfaisant $q(x) < p(x)$, pour tout $x \in \bar{\Omega}$. On suppose que $f : \Omega \times \mathbf{R} \to \mathbf{R}$ est une fonction de Caratheodory qui satisfait la condition de croissance*

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1631-073X/$ – see front matter
doi:10.1016/j.crma.2009.04.022

$$\left|f(x,s)\right| \leqslant c_1|s|^{\frac{q(x)}{q'(x)}} + a(x), \quad \textit{p.p. sur } \Omega \textit{ et pour tout } s \in \mathbf{R}, \tag{1}$$

où $c_1 = \text{cte.} > 0$, $\frac{1}{q(x)} + \frac{1}{q'(x)} = 1$ *pour tout* $x \in \bar{\Omega}$ *et* $a \in L^{q'(\cdot)}(\Omega)$, $a(x) \geqslant 0$ *p.p. sur* Ω. *Alors, le problème de Dirichlet*

$$\begin{aligned} &-\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) = f(x,u), \quad x \in \Omega, \qquad (\mathcal{P})\\ &u = 0, \quad x \in \partial\Omega, \end{aligned}$$

possède une solution faible dans $W_0^{1,p(\cdot)}(\Omega)$.

Dans cette Note nous montrons que le résultat du Théorème 1 peut être obtenu en utilisant une méthode topologique, basée sur les propriétés fondamentales du degré de Leray–Schauder. Les idées de la démonstration sont les suivantes : (i) les inégalités

$$1 < q(x) < p(x) < p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)}, & p(x) < N,\\ +\infty, & p(x) \geqslant N, \end{cases}$$

entraînent la compacité de l'immersion de $W_0^{1,p(\cdot)}(\Omega)$ dans $L^{q(\cdot)}(\Omega)$ (cf. [3]) ; (ii) la condition (1), imposée à la fonction de Caratheodory f, implique que l'opérateur de Nemytsckij engendré par f, $(N_f u)(x) = f(x,u(x))$, est bien défini de $L^{q(\cdot)}(\Omega)$ à valeurs dans $L^{q'(\cdot)}(\Omega)$ et, aussi que $N_f : L^{q(\cdot)}(\Omega) \to L^{q'(\cdot)}(\Omega)$ est continu et borné ; (iii) pour montrer que le problème $(\mathcal{P})$ possède une solution faible dans $W_0^{1,p(\cdot)}(\Omega)$ il suffit de montrer qu'il existe $u \in W_0^{1,p(\cdot)}(\Omega)$ qui satisfait $-\Delta_{p(\cdot)}u = (i^* N_f i)u$ où i est l'injection compacte de $W_0^{1,p(\cdot)}(\Omega)$ dans $L^{q(\cdot)}(\Omega)$ et $i^* : L^{q'(\cdot)}(\Omega) \to (W_0^{1,p(\cdot)}(\Omega))^*$, $i^*v = v \circ i$ pour tout $v \in L^{q'(\cdot)}(\Omega)$, représente son adjointe ; (iv) puisque $(i^* N_f i) : W_0^{1,p}(\Omega) \to (W_0^{1,p(\cdot)}(\Omega))^*$ est compact et $-\Delta_{p(\cdot)} : W_0^{1,p}(\Omega) \to (W_0^{1,p(\cdot)}(\Omega))^*$ est un homéomorphisme, le problème est réduit à montrer que l'opérateur compact $K = (-\Delta_{p(\cdot)})^{-1}(i^* N_f i) : W_0^{1,p}(\Omega) \to W_0^{1,p(\cdot)}(\Omega)$ posséde un point fixe ; (v) on démontre l'existence d'un point fixe pour K en montrant que l'ensemble

$$\mathcal{S} = \left\{u \in W_0^{1,p(\cdot)}(\Omega) \;\middle|\; \exists t \in [0,1] \text{ t.q. } u = tKu\right\}$$

est borné, via le théorème de Leray–Schauder–Schaefer.

1. The main result

By using a variational method, Fan and Zhang [4] have proved the following result:

Theorem 1. *Let* $\Omega \subset \mathbf{R}^N$, $N \geqslant 2$, *be a smooth bounded domain and* $p, q \in \mathcal{C}_+(\bar{\Omega}) = \{h \in \mathcal{C}(\bar{\Omega}) \mid h(x) > 1$ *for all* $x \in \bar{\Omega}\}$ *be such that* $q(x) < p(x)$ *for all* $x \in \bar{\Omega}$. *Assume that* $f : \Omega \times \mathbf{R} \to \mathbf{R}$ *is a Caratheodory function which satisfies the growth condition*

$$\left|f(x,s)\right| \leqslant c_1|s|^{\frac{q(x)}{q'(x)}} + a(x), \quad \textit{for a.e. } x \in \Omega \textit{ and all } s \in \mathbf{R}, \tag{1}$$

with $c_1 = \text{const.} > 0$, $\frac{1}{q(x)} + \frac{1}{q'(x)} = 1$ *for all* $x \in \bar{\Omega}$ *and* $a \in L^{q'(\cdot)}(\Omega)$, $a(x) \geqslant 0$ *for a.e.* $x \in \Omega$.

Then, Dirichlet problem $(\mathcal{P})$ *has a weak solution in* $W_0^{1,p(\cdot)}(\Omega)$.

We are going to show that a topological method, based on the fundamental properties of the Leray–Schauder degree, may be used in proving the above theorem.

Let us first recall that

$$L^{p(\cdot)}(\Omega) = \left\{u \;\middle|\; u \text{ is (Lebesque) measurable real function in } \Omega \text{ and } \rho_{p(\cdot)}(u) = \int_\Omega \left|u(x)\right|^{p(x)} \mathrm{d}x < \infty\right\}.$$

Endowed with the norm

$$\|u\|_{p(\cdot)} = \inf\left\{\lambda > 0 \mid \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} \mathrm{d}x \leqslant 1\right\}, \quad \text{for all } u \in L^{p(\cdot)}(\Omega),$$

$(L^{p(\cdot)}(\Omega), \|\ \|_{p(\cdot)})$ is a separable, uniform convex Banach space and its conjugate space is $(L^{p'(\cdot)}(\Omega), \|\ \|_{p'(\cdot)})$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ for all $x \in \bar{\Omega}$ (cf. [4, Proposition 2.1]). Write $h^+ = \max_{x\in\bar{\Omega}} h(x)$, $h^- = \min_{x\in\bar{\Omega}} h(x)$ for any $h \in \mathcal{C}(\bar{\Omega})$.

The next proposition illuminates the close relation between the norm $\|\ \|_{p(\cdot)}$ a nd the convex modular $\rho_{p(\cdot)}$:

Proposition 1. *(See [5, Theorems 1.2 and 1.3].) One has*:

(a) $\|u\|_{p(\cdot)} > 1 \implies \|u\|_{p(\cdot)}^{p^-} \leqslant \rho_{p(\cdot)}(u) \leqslant \|u\|_{p(\cdot)}^{p^+}$;
(b) $\|u\|_{p(\cdot)} < 1 \implies \|u\|_{p(\cdot)}^{p^-} \geqslant \rho_{p(\cdot)}(u) \geqslant \|u\|_{p(\cdot)}^{p^+}$;
(c) $\|u\|_{p(\cdot)} = a > 0 \iff \rho_{p(\cdot)}(\frac{u}{a}) = 1$;
(d) $\|u\|_{p(\cdot)} < 1 (= 1; > 1) \iff \rho_{p(\cdot)}(u) < 1 (= 1; > 1)$.

Remark 1. From Proposition 1, one easily derives that

(e) $\|u\|_{p(\cdot)} < \rho_{p(\cdot)}(u) + 1$,
(f) $\rho_{p(\cdot)}(u) < \|u\|_{p(\cdot)}^{p^+} + 1$,

for all $u \in L^{p(\cdot)}(\Omega)$.

The space $W^{1,p(\cdot)}(\Omega)$ is defined by

$$W^{1,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) \mid |\nabla u| \in L^{p(\cdot)}(\Omega)\}, \quad |\nabla u|^2 = \sum_{i=1}^N \left(\frac{\partial u}{\partial x_i}\right)^2$$

and it is endowed with the norm

$$\|u\| = \|u\|_{p(\cdot)} + \big\||\nabla u|\big\|_{p(\cdot)}, \quad \text{for all } u \in W^{1,p(\cdot)}(\Omega).$$

We define $W_0^{1,p(\cdot)}(\Omega)$ as the closure of $\mathcal{C}_0^\infty(\Omega)$ in $(W^{1,p(\cdot)}(\Omega), \|\ \|)$. Due to Poincaré's inequality

$$\|u\|_{p(\cdot)} \leqslant c\big\||\nabla u|\big\|_{p(\cdot)}, \quad \text{for all } u \in W^{1,p(\cdot)}(\Omega),$$

we infer that $\|u\|$ and $\|u\|_{1,p(\cdot)} = \||\nabla u|\|_{p(\cdot)}$ are equivalent norms on $W_0^{1,p(\cdot)}(\Omega)$. One has (Fan and Zhao [5]): $(W_0^{1,p(\cdot)}(\Omega), \|\ \|_{1,p(\cdot)})$ is a separable reflexive Banach space, compactly embedded in $(L^{q(\cdot)}(\Omega), \|\ \|_{q(\cdot)})$ for any $q \in \mathcal{C}_+(\bar{\Omega})$ which satisfies

$$q(x) < p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)}, & p(x) < N, \\ +\infty, & p(x) \geqslant N. \end{cases}$$

In what follows, $W_0^{1,p(\cdot)}(\Omega)$ will be considered as endowed with the norm $\|\ \|_{1,p(\cdot)}$ and we will often write $W_0^{1,p(\cdot)}(\Omega)$ and $L^{p(\cdot)}(\Omega)$ instead of $(W_0^{1,p(\cdot)}(\Omega), \|\ \|_{1,p(\cdot)})$ and $(L^{p(\cdot)}(\Omega), \|\ \|_{p(\cdot)})$, respectively.

Concerning problem $(\mathcal{P})$, let us first observe that, for all $u \in W_0^{1,p(\cdot)}(\Omega)$ and $i = 1, 2, \ldots, N$, $|\nabla u|^{p(\cdot)-2} \frac{\partial u}{\partial x_i} \in L^{p'(\cdot)}(\Omega)$. According to the characterization of linear and continuous functionals on $W_0^{1,p(\cdot)}(\Omega)$ given by Hudzik [6], we deduce from this, that the operator

$$u \in W_0^{1,p(\cdot)}(\Omega) \to -\Delta_{p(\cdot)} u = -\frac{\partial}{\partial x_i}\left(|\nabla u|^{p(\cdot)-2} \frac{\partial u}{\partial x_i}\right)$$

is well defined from $W_0^{1,p(\cdot)}(\Omega)$ into its dual $(W_0^{1,p(\cdot)}(\Omega))^*$ and

$$\langle -\Delta_{p(\cdot)}u, v\rangle_{W_0^{1,p(\cdot)}(\Omega),(W_0^{1,p(\cdot)}(\Omega))^*} = \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v \,\mathrm{d}x, \quad \text{for all } v \in W_0^{1,p(\cdot)}(\Omega). \tag{2}$$

Moreover, by Theorem 3.1 in [4], $-\Delta_{p(\cdot)}$ is a homeomorphism of $W_0^{1,p(\cdot)}(\Omega)$ onto $(W_0^{1,p(\cdot)}(\Omega))^*$.

Secondly, since $q(x) < p(x) < p^*(x)$ for all $x \in \bar{\Omega}$, $W_0^{1,p(\cdot)}(\Omega)$ is compactly embedded in $L^{q(\cdot)}(\Omega)$. Denote by i the compact injection of $W_0^{1,p(\cdot)}(\Omega)$ in $L^{q(\cdot)}(\Omega)$ and by $i^* : L^{q'(\cdot)}(\Omega) \to (W_0^{1,p(\cdot)}(\Omega))^*$, $i^*v = v \circ i$ for all $v \in L^{q'(\cdot)}(\Omega)$, its adjoint.

Finally, since the Caratheodory function f satisfies (1), the Nemytsckij operator N_f generated by f, $(N_f u)(x) = f(x, u(x))$, is well defined from $L^{q(\cdot)}(\Omega)$ into $L^{q'(\cdot)}(\Omega)$, continuous and bounded ([4, Proposition 2.2]). In order to prove that problem $(\mathcal{P})$ has a weak solution in $W_0^{1,p(\cdot)}(\Omega)$ it is sufficient to prove that the equation

$$-\Delta_{p(\cdot)}u = (i^* N_f i)u \tag{3}$$

has a solution in $W_0^{1,p(\cdot)}(\Omega)$.

Indeed, if $u \in W_0^{1,p(\cdot)}(\Omega)$ satisfies (3) then, for all $v \in W_0^{1,p(\cdot)}(\Omega)$, one has

$$\langle -\Delta_{p(\cdot)}u, v\rangle_{W_0^{1,p(\cdot)}(\Omega),(W_0^{1,p(\cdot)}(\Omega))^*} = \langle (i^* N_f i)u, v\rangle_{W_0^{1,p(\cdot)}(\Omega),(W_0^{1,p(\cdot)}(\Omega))^*} = \langle N_f(iu), iv\rangle_{L^{q(\cdot)}(\Omega),L^{q'(\cdot)}(\Omega)},$$

which rewrites as

$$\int_\Omega |\nabla u|^{p(x)-2} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\,\mathrm{d}x = \int_\Omega f\big(x, u(x)\big)v(x)\,\mathrm{d}x$$

and tells us that u is a weak solution in $W_0^{1,p(\cdot)}(\Omega)$ to problem $(\mathcal{P})$.

Since $-\Delta_{p(\cdot)}$ is a homeomorphism of $W_0^{1,p(\cdot)}(\Omega)$ onto $(W_0^{1,p(\cdot)}(\Omega))^*$, (3) may be equivalently written as

$$u = (-\Delta_{p(\cdot)})^{-1}\big[(i^* N_f i)u\big]. \tag{4}$$

Thus, proving that problem $(\mathcal{P})$ has a weak solution in $W_0^{1,p(\cdot)}(\Omega)$ reduces to proving that the compact operator

$$K = (-\Delta_{p(\cdot)})^{-1}(i^* N_f i) : W_0^{1,p(\cdot)}(\Omega) \to W_0^{1,p(\cdot)}(\Omega)$$

has a fixed point. By a classical result of Leray, Schauder and Schaefer, a sufficient condition for K to have a fixed point is that a constant $R > 0$ exists such that

$$\mathcal{S} = \big\{u \in W_0^{1,p(\cdot)}(\Omega) \,\big|\, u = tKu \quad \text{for some } t \in [0,1]\big\} \subset \mathcal{B}(0,R).$$

Since, for $t = 0$, the only solution of equation $u = tKu$ is $u = 0$, it is enough to show that there exists a constant $R > 0$ such that, any $u \in W_0^{1,p(\cdot)}(\Omega)$ which satisfies

$$u = t(-\Delta_{p(\cdot)})^{-1}\big[(i^* N_f i)u\big] \tag{5}$$

for some $t \in (0,1]$, belongs to $\mathcal{B}(0,R)$.

Indeed, if $u \in W_0^{1,p(\cdot)}(\Omega)$ satisfies (5) for some $t \in (0,1]$ then one has:

$$\left\langle -\Delta_{p(\cdot)}\left(\frac{u}{t}\right), \left(\frac{u}{t}\right)\right\rangle_{W_0^{1,p(\cdot)}(\Omega),(W_0^{1,p(\cdot)}(\Omega))^*} = \frac{1}{t}\big\langle (i^* N_f i)u, u\big\rangle_{W_0^{1,p(\cdot)}(\Omega),(W_0^{1,p(\cdot)}(\Omega))^*}. \tag{6}$$

Since

$$\left\langle -\Delta_{p(\cdot)}\left(\frac{u}{t}\right), \left(\frac{u}{t}\right)\right\rangle_{W_0^{1,p(\cdot)}(\Omega),(W_0^{1,p(\cdot)}(\Omega))^*} = \int_\Omega \left|\nabla\left(\frac{u}{t}\right)\right|^{p(x)} \mathrm{d}x \geqslant \frac{1}{t^{p^-}}\int_\Omega |\nabla u|^{p(x)}\,\mathrm{d}x = \frac{1}{t^{p^-}}\rho_{p(\cdot)}\big(|\nabla u|\big),$$

it follows from (6) that

$$\rho_{p(\cdot)}\big(|\nabla u|\big) \leqslant t^{p^- -1}\big\langle (i^* N_f i)u, u\big\rangle_{W_0^{1,p(\cdot)}(\Omega),(W_0^{1,p(\cdot)}(\Omega))^*} \leqslant \big\langle (i^* N_f i)u, u\big\rangle_{W_0^{1,p(\cdot)}(\Omega),(W_0^{1,p(\cdot)}(\Omega))^*} \leqslant \|i^*\| \big\|N_f(iu)\big\|_{q'(\cdot)} \|u\|_{1,p(\cdot)}. \tag{7}$$

In order to estimate $\|N_f(iu)\|_{q'(\cdot)}$, first we prove that

$$\|N_f v\|_{q'(\cdot)} < c_1 \|v\|_{q(\cdot)}^{q^+-1} + \big(2c_1 + \|a\|_{q'(\cdot)}\big), \quad \text{for all } v \in L^{q(\cdot)}(\Omega). \tag{8}$$

Indeed, since

$$\big|(N_f v)(x)\big| = \big|f(x, v(x)\big| \leqslant c_1 \big|v(x)\big|^{\frac{q(x)}{q'(x)}} + a(x),$$

we deduce that

$$\|N_f v\|_{q'(\cdot)} \leqslant \Big\|c_1 \big|v(\cdot)\big|^{\frac{q(\cdot)}{q'(\cdot)}} + a(\cdot)\Big\|_{q'(\cdot)} \leqslant c_1 \Big\|\big|v(\cdot)\big|^{\frac{q(\cdot)}{q'(\cdot)}}\Big\|_{q'(\cdot)} + \big\|a(\cdot)\big\|_{q'(\cdot)}. \tag{9}$$

Let us prove that

$$\Big\|\big|v(\cdot)\big|^{\frac{q(\cdot)}{q'(\cdot)}}\Big\|_{q'(\cdot)} < \|v\|_{q(\cdot)}^{q^+-1} + 2. \tag{10}$$

Indeed, one has:

$$\text{(i)} \quad \|v\|_{q(\cdot)} \geqslant 1 \quad \Longrightarrow \quad \Big\|\big|v(\cdot)\big|^{\frac{q(\cdot)}{q'(\cdot)}}\Big\|_{q'(\cdot)} \leqslant \|v\|_{q(\cdot)}^{q^+-1}. \tag{11}$$

This is seen as follows: According to Proposition 1(d), proving (i) it is equivalent to proving that $\|v\|_{q(\cdot)} \geqslant 1$ implies

$$\rho_{q'(\cdot)}\left(\frac{|v(\cdot)|^{\frac{q(\cdot)}{q'(\cdot)}}}{\|v\|_{q(\cdot)}^{q^+-1}}\right) = \int\limits_\Omega \frac{|v(x)|^{q(x)}}{\|v\|_{q(\cdot)}^{(q^+-1)q'(x)}}\,\mathrm{d}x \leqslant 1. \tag{12}$$

This inequality is justified as follows. Since $\|v\|_{q(\cdot)} \geqslant 1$ and

$$(q^+ - 1)q'(x) - q(x) = q^+ q'(x) - \big(q(x) + q'(x)\big) = q^+ q'(x) - q(x)q'(x) = q'(x)\big(q^+ - q(x)\big) \geqslant 0,$$

we infer that

$$\frac{|v(x)|^{q(x)}}{\|v\|_{q(\cdot)}^{(q^+-1)q'(x)}} = \frac{|v(x)|^{q(x)}}{\|v\|_{q(\cdot)}^{q(x)}} \cdot \frac{1}{\|v\|_{q(\cdot)}^{(q^+-1)q'(x)-q(x)}} \leqslant \frac{|v(x)|^{q(x)}}{\|v\|_{q(\cdot)}^{q(x)}}. \tag{13}$$

From (12), (13) and Proposition 1(c) we derive that

$$\rho_{q'(\cdot)}\left(\frac{|v(\cdot)|^{\frac{q(\cdot)}{q'(\cdot)}}}{\|v\|_{q(\cdot)}^{q^+-1}}\right) \leqslant \rho_{q(\cdot)}\left(\frac{v}{\|v\|_{q(\cdot)}}\right) = 1$$

and the prove of (i) is complete.

$$\text{(ii)} \quad \|v\|_{q(\cdot)} < 1 \quad \Longrightarrow \quad \Big\|\big|v(\cdot)\big|^{\frac{q(\cdot)}{q'(\cdot)}}\Big\|_{q'(\cdot)} < 2. \tag{14}$$

Indeed, by Remark 1(e) and Proposition 1(d), one has:

$$\Big\|\big|v(\cdot)\big|^{\frac{q(\cdot)}{q'(\cdot)}}\Big\|_{q'(\cdot)} < \rho_{q'(\cdot)}\Big(\big|v(\cdot)\big|^{\frac{q(\cdot)}{q'(\cdot)}}\Big) + 1 = \rho_{q(\cdot)}(v) + 1 < 1 + 1 = 2.$$

Clearly, (10) is a consequence of (11) and (14) and (8) is a consequence of (9) and (10).

By writing (8) for $v = iu$, $u \in W_0^{1,p(\cdot)}(\Omega)$, we get:

$$\big\|N_f(iu)\big\|_{q'(\cdot)} < c_1 \|i\|^{q^+-1} \|u\|_{1,p(\cdot)}^{q^+-1} + \big(2c_1 + \|a\|_{q'(\cdot)}\big). \tag{15}$$

In particular, if $u \in W_0^{1,p(\cdot)}(\Omega)$ and satisfies (5) for some $t \in (0, 1]$, we derive from (15) and (7) that

$$\rho_{p(\cdot)}\big(|\nabla u|\big) < \big(k_1\|u\|_{1,p(\cdot)}^{q^+-1} + k_2\big)\|u\|_{1,p(\cdot)} \tag{16}$$

with $k_1 = c_1\|i\|^{q^+}$, $k_2 = (2c_1 + \|a\|_{q'(\cdot)})\|i\|$.

We claim that from (16) it follows the boundedness of the set $\mathcal{S}$.

Assume the contrary: there exists a sequence $(u_n) \subset \mathcal{S}$ such that $\|u_n\|_{1,p(\cdot)} \to \infty$ as $n \to \infty$. Then, for every u_n with $\|u_n\|_{1,p(\cdot)} = \||\nabla u|\|_{p(\cdot)} > 1$, we would have (Proposition 1(a))

$$\rho_{p(\cdot)}\big(|\nabla u_n|\big) \geqslant \big\||\nabla u_n|\big\|_{p(\cdot)}^{p^-} = \|u_n\|_{1,p(\cdot)}^{p^-} \tag{17}$$

and, from (17) and (16),

$$\|u_n\|_{1,p(\cdot)}^{p^--1} < k_1\|u_n\|_{1,p(\cdot)}^{q^+-1} + k_2.$$

Since $q^+ < p^-$, this last inequality implies the boundedness of (u_n), a contradiction.

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