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Uniform bound and a non-existence result for Lichnerowicz equation in the whole n-space ☆

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Abstract

In this Note, we give a uniform bound and a non-existence result for positive solutions to the Lichnerowicz equation in $\mathbf{R}^n$. In particular, we show that positive smooth solutions to:

$$\Delta u + f(u) = 0, \quad u > 0, \quad \text{in } \mathbf{R}^n$$

where

$$f(u) = u^{-p-1} - u^{p-1},$$

are uniformly bounded. ***To cite this article: L. Ma, X. Xu, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***
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Résumé

Une estimation uniforme et un résultat de non-existence pour l'équation de Lichnerowicz sur *n*-espace. Dans cette Note, nous donnons une estimation uniforme et un résultat de non-existence pour les solutions positives de l'équation de Lichnerowicz sur $\mathbf{R}^n$. En particulier, nous montrons que les solutions positives régulières de :

$$\Delta u + f(u) = 0, \quad u > 0, \quad \text{dans } \mathbf{R}^n$$

où

$$f(u) = u^{-p-1} - u^{p-1},$$

sont bornées. ***Pour citer cet article : L. Ma, X. Xu, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***
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1. Introduction

In the Einstein-scalar field theory one has the Lichnerowicz equation on a Riemannian manifold (M, γ) of dimension $n \geqslant 3$ (see [2,3,6]). The aim of this paper is to give some results for positive solutions to this equation in the whole Euclidean space.

Given a smooth symmetric 2-tensor σ, a smooth vector field W, and a triple data (π, τ, φ) of smooth functions on M. Set

$$c_n = \frac{n-2}{4(n-1)}, \qquad p = \frac{2n}{n-2},$$

and let

$$R_{\gamma,\varphi} = c_n\big(R(\gamma) - |\nabla\varphi|_\gamma^2\big), \qquad A_{\gamma,W,\pi} = c_n\big(|\sigma + DW|_\gamma^2 + \pi^2\big)$$

and

$$B_{\tau,\varphi} = c_n\left(\frac{n-1}{n}\tau^2 - V(\varphi)\right)$$

where $V:\mathbf{R}\to\mathbf{R}$ is a given smooth function and $R(\gamma)$ is the scalar curvature function of γ. Then the Lichnerowicz equation for the Einstein-scalar conformal data $(\gamma, \sigma, \pi, \tau, \varphi)$ with the given vector field W is

$$\Delta_\gamma u - R_{\gamma,\varphi}u + A_{\gamma,W,\pi}u^{-p-1} - B_{\tau,\varphi}u^{p-1} = 0, \quad u > 0, \tag{1}$$

where Δ_γ is the Laplacian operator of γ. We use the convention that $\Delta_\gamma u = u''$ on the real line $\mathbf{R}$. Note that $A_{\gamma,W,\pi} \geqslant 0$. This equation is closely related to the Yamabe problem and the prescribing scalar curvature problems (see [1,7,8]).

We shall consider a special case when $(M, \gamma) = \mathbf{R}^n$ is the standard Euclidean space with radial symmetry data $(\sigma, \pi, \tau, \varphi)$. In this case, we can simply rewrite the equation in the following form

$$\Delta u + R(x)u + A(x)u^{-p-1} + B(x)u^{p-1} = 0, \quad u > 0, \quad \text{on } \mathbf{R}^n \tag{2}$$

where $R(x) \geqslant 0$, $A(x) \geqslant 0$, and $B(x)$ are given smooth functions of $x \in R^n$.

Theorem 1. *Suppose that $A := A(x) \geqslant 0$, $B := B(x) \geqslant 0$, and $R(x) \geqslant 0$. Let $\beta = \frac{p+1}{2p}$. Assume that*

$$\int_0^{+\infty} \mathrm{d}r\left(r^{1-n}\int_{B_r(0)} A^{1-\beta}B^\beta\,\mathrm{d}x\right) = +\infty. \tag{3}$$

Then there exists no positive solution to (2).

Note that $\beta = \frac{3n-2}{4n}$, so the condition (3) can be written as

$$\int_0^{+\infty} \mathrm{d}r\left(r^{1-n}\int_{B_r(0)} A(x)^{\frac{n+2}{4n}}B(x)^{\frac{3n-2}{4n}}\,\mathrm{d}x\right) = +\infty.$$

As a particular example, we note that when $A^{1-\beta}B^\beta \geqslant C > 0$ for some positive constant $C > 0$, there exists no positive solution to (2).

This result may be extended to other case (see Theorem 3 in next section).

We also have the following uniform bound for any positive solution to (2).

Proposition 2. *Assume that $R(x) = 0$ and $A(x) = 1$ is a positive constant and $B(x) = -B$ is a negative constant in* (2). *Then any positive solution to* (2) *is uniformly bounded.*

In a recent paper, O. Druet and E. Hebey [4] have proved a very interesting result which says that for Lichnerowicz equation on a compact Riemannian manifold, the stability holds true when the dimension n is such that $n \leqslant 5$ and fails to hold in general when $n \geqslant 6$.

2. Non-existence results

In this section we prove Theorem 1.

Recall our assumption that $B(x) \geqslant 0$ and $R(r) \geqslant 0$. We remark that for each fixed $x \in R^n$,

$$A(x)X^{-p-1} + B(x)X^{p-1}$$

is a convex function in X.

Proof of Theorem 1. Let $\bar{u} := \bar{u}(r)$ be the average of $u(x)$ on the sphere $S_r^{n-1}(0)$ of radius r.

Note that taking this average operation and using Jensen's inequality to Eq. (2) we have

$$-\bar{u}'' - \frac{n-1}{r}\bar{u}' \geqslant \overline{R(x)u} + \overline{A(x)u^{-p-1} + B(x)u^{p-1}}. \tag{4}$$

Using the Holder inequality to the right side of (4), we have

$$\overline{A(x)u^{-p-1} + B(x)u^{p-1}} \geqslant \overline{A^{1-\beta}B^{\beta}}$$

where

$$\beta = \frac{p+1}{2p}.$$

Then we have

$$-\left(r^{n-1}\bar{u}'\right)' \geqslant r^{n-1}\left(\overline{R(x)u} + \overline{A^{1-\beta}B^{\beta}}\right),$$

which implies that

$$-r^{n-1}\bar{u}' \geqslant \int_{B_r(0)} A^{1-\beta}B^{\beta}\,\mathrm{d}x + \int_{B_r(0)} Ru$$

after an integration. Dividing both side by r^{n-1} and integrating this inequality over $[0, r_0]$, we have

$$\bar{u}(0) - \bar{u}(r_0) \geqslant \int_0^{r_0} \mathrm{d}r\left(r^{1-n}\int_{B_r(0)} A^{1-\beta}B^{\beta}\,\mathrm{d}x\right) + \int_0^{r_0} r^{1-n}\int_{B_r(0)} Ru.$$

Sending $r_0 \to \infty$ we have

$$\bar{u}(0) \geqslant \int_0^{\infty} \mathrm{d}r\left(r^{1-n}\int_0^{r} \tau^{n-1}A^{1-\beta}B^{\beta}\,\mathrm{d}\tau\right),$$

which is impossible by our assumption that

$$\int_0^{+\infty} \mathrm{d}r\left(r^{1-n}\int_{B_r(0)} A^{1-\beta}B^{\beta}\,\mathrm{d}x\right) = +\infty.$$

Then the proof of Theorem 1 is complete.

We remark that from our proof above, we use the interaction between A and B. If we use the interaction between R and A, we can have the following result by the same argument.

Theorem 3. *Suppose that $A := A(x) \geqslant 0$, $B(x) \geqslant 0$, and $R := R(x) \geqslant 0$. Let $\beta = \frac{p+1}{2p}$. Assume that*

$$\int_0^{+\infty} \mathrm{d}r\left(r^{1-n}\int_{B_r(0)} A(x)^{\frac{n-2}{4(n-1)}}R(x)^{\frac{3n-2}{4(n-1)}}\,\mathrm{d}x\right) = +\infty.$$

Then there exists no positive solution to (2).

3. Proof of Proposition 2

In this section, we assume that $R(r)=0$ and $A(r)=1$ is a positive constant and $B(r)=-B$ is a negative constant in (2). Then we may reduce (2) into the following form:

$$\Delta u + f(u) = 0, \quad u > 0, \quad \text{on } \mathbf{R}^n \tag{5}$$

where

$$f(u) = u^{-p-1} - Bu^{p-1}.$$

Denote by B_R any ball of radius $R > 0$ in $\mathbf{R}^n$.

We shall use a trick used in [5]. We look for a positive radial super-solution $v = v(r)$ to (5) in the ball B_R with the positive infinity boundary condition. This is equivalent to finding $v = v(r) > 0$ such that

$$\begin{cases} \Delta v + f(v) \leqslant 0, & \text{in } B_R, \\ v = +\infty, & \text{on } \partial B_R. \end{cases}$$

Note that

$$f' = -(p+1)u^{-p} - B(p-1)u^{p-2} < 0$$

for $u > 0$. Then the comparison lemma is true for (5) in the ball B_R. Hence, we have

$$u(x) \leqslant v(r), \quad \text{in } B_R.$$

From this we know that u is uniformly bounded in $\mathbf{R}^n$.

Let $v(r) = (R^2 - r^2)^{-\alpha}$ for large $\alpha > 1$ and small $R \ll 1$. By direct computation, we know that v is the right super-solution $v = v(r)$ to (5) in the ball B_R with positive infinity boundary condition. Hence

$$u(x) \leqslant 2^{\alpha} R^{-2\alpha}, \quad \text{in } B_{R/2}.$$

This proves our Proposition 2.

It is clear that our argument can be generalized to treat positive solutions to the following equation:

$$\Delta u + A(x)u^{-p-1} - Bu^{p-1} = 0, \quad \text{in } \mathbf{R}^n,$$

where $A(x)$ is a smooth uniformly bounded function in $\mathbf{R}^n$. It is an open question if the Liouville type theorem is true for positive solutions to (5).

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