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Quasi-frames of translates ☆

Shahaf Nitzan, Alexander Olevskii

School of Mathematical Sciences, Tel Aviv University, Ramat-Aviv, Israel 69978

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Abstract

We construct uniformly discrete, and even sparse, sequences of translates $\{g(t-\lambda)\}$ of a single function which have the following frame-type approximation property: for every $q>2$ there exists $C(q)$ such that every function $f\in L^2(\mathbb{R})$ can be approximated with arbitrary small L^2-error by a linear combination $\sum c_\lambda g(t-\lambda)$ satisfying the l_q-estimate of the coefficients:

$$\big\|\{c_\lambda\}\big\|_{l_q}\leqslant C(q)\|f\|.$$

This cannot be done for $q=2$, according to a result of Christensen, Deng and Heil. ***To cite this article: S. Nitzan, A. Olevskii, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Systèmes de translatées proches des *frames*. Nous construisons une suite réelle Λ uniformément discrète (de pas >0) et même lacunaire, et une fonction $g\in L^2(\mathbb{R})$, telles que le système des translatées $\{g(t-\lambda)\}(\lambda\in\Lambda)$ soit un "*quasi-frame*" au sens suivant : pour tout $q>2$ il existe $C(q)>0$ tel que toute fonction $f\in L^2(\mathbb{R})$ est approchable dans $L^2(\mathbb{R})$ par des combinaisons linéaires $\sum c_\lambda g(t-\lambda)$ vérifiant $(\sum|c_\lambda|^q)^{1/q}\leqslant C(q)\|f\|$. Cela est impossible quand $q=2$, selon un résultat de Christensen, Deng et Heil. ***Pour citer cet article : S. Nitzan, A. Olevskii, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

1. Introduction

1.1. Let Λ be a uniformly discrete (u.d.) set of real numbers:

$$\gamma(\Lambda):=\inf_{\lambda,\lambda'\in\Lambda,\lambda\neq\lambda'}|\lambda-\lambda'|>0. \tag{1}$$

Given a function $g\in L^2(\mathbb{R})$, consider the family of translates

$$\big\{g(t-\lambda)\big\}_{\lambda\in\Lambda}. \tag{2}$$

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E-mail addresses: nitzansi@post.tau.ac.il (S. Nitzan), olevskii@post.tau.ac.il (A. Olevskii).

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When $\Lambda = \mathbb{Z}$, it is well known that this family cannot be complete in $L^2(\mathbb{R})$. It was conjectured that the same was true for every u.d. set Λ (see for example [9], where even a stronger conjecture related to Gabor-type systems was discussed). However, this is not the case:

Theorem A. *(See [6].) Let $\Lambda = \{\lambda_n\}_{n\in\mathbb{Z}}$ be any "almost integer" spectrum*:

$$\lambda_n = n + \alpha_n, \quad n \in \mathbb{Z}, \qquad 0 < |\alpha_n| \to 0 \quad \bigl(|n| \to \infty\bigr).$$

Then there is a "generator" g such that family (2) *is complete in $L^2(\mathbb{R})$.*

See also paper [7], which considers sparse complete systems of translates.

One may want to construct a u.d. or even a sparse set Λ for which there is a family (2) satisfying a stronger property than just completeness. Observe, however, that no family (2) can be a frame in $L^2(\mathbb{R})$ (see [1]).

1.2. In [5] we introduced an intermediate property between completeness and frame, which we formulate below in a slightly different form:

Definition 1. We say that a system of vectors $\{u_n\}$ in a Hilbert space H is a (QF)-system if the following two conditions are fulfilled:

(i) For every $q > 2$ there is a constant $C(q)$ such that for every $f \in H$ and every $\epsilon > 0$ one can find a finite linear combination $Q = \sum c_n u_n$ satisfying $\|f - Q\| < \epsilon$ and $\|\{c_n\}\|_{l_q} \leqslant C(q)\|f\|$.
(ii) (Bessel's inequality) We have $\|\sum a_n u_n\| \leqslant C\|\{a_n\}\|_{l_2}$, for every finite sequence $\{a_n\}$.

Approximation property (i) means "completeness with l_q-estimate of coefficients". If this condition holds for $q = 2$, Definition 1 becomes identical with the usual definition of a frame. One may therefore regard (QF)-systems as a sort of "quasi-frames".

1.3. The main result of this Note is the following:

Theorem 1. *There are a u.d. sequence $\Lambda = \{\lambda_1 < \lambda_2 < \cdots\} \subset \mathbb{R}^+$ and a function $g \in L^2(\mathbb{R})$ such that the system* (2) *is a* (*QF*)*-system for $L^2(\mathbb{R})$. Moreover, for every positive sequence $\epsilon(n) = o(1)$, one can choose Λ such that*

$$\lambda_{n+1} > \bigl(1 + \epsilon(n)\bigr)\lambda_n. \tag{3}$$

Clearly, if the ϵ_n have a slow decay, then the gaps in the spectrum Λ grow "almost exponentially". This condition is sharp.

2. Proof

2.1. Similarly to [6], one can re-formulate the main result in an equivalent form:

Theorem 2. *Given a decreasing sequence $\epsilon(n) = o(1)$ there are a weight w and a positive sequence Λ such that*:

(i) *Λ satisfies* (3).
(ii) *The system $E(\Lambda) := \{e^{i\lambda x}\}_{\lambda\in\Lambda}$ is a* (*QF*)*-system in $L^2(w, \mathbb{R})$.*

By a "weight" we understand an a.e. positive integrable function. The connection between weights and generators is given by $w(x) = |\hat{g}(x)|^2$. We will sketch the proof of Theorem 2.

2.2. In what follows, we denote by I a finite interval on $\mathbb{R}$, by $\|\mathbf{c}\|_q$ the l_q-norm of a sequence $\mathbf{c} = \{c_k\}$ and by mE the measure of E.

The first lemma is about "analytic unity", see [2], p. 102 and [3] for the proof.

Lemma 1. *For every interval I and numbers $q > 2$ and $\epsilon > 0$, one can find a trigonometric polynomial $A(x) = \sum_{k=1}^{K} a_k e^{ikx}$ such that $\|\mathbf{a}\|_q < \epsilon$ and*

$$m\{x \in I\colon |A(x) - 1| > \epsilon\} < \epsilon.$$

Lemma 2. *For every function $f \in L^2(I)$ and number $\xi > 0$, one can find a trigonometric polynomial $B(x) = \sum_{n=1}^{N} b_n e^{i\beta(n)x}$, such that $|\beta(n) - n| < 1$, $n = 1, \ldots, N$, and*

$$m\{x \in I\colon |f(x) - B(x)| > \xi\} < \xi.$$

This lemma can be easily deduced from the result in [4].

Lemma 3. *For every $q > 2$, $\delta > 0$ and $f \in L^2(I)$, there is a number $\mu > 0$ such that for every integer $d > 0$ there is a trigonometric polynomial $Q(x) = \sum c_m e^{i\lambda_m x}$, $\|\mathbf{c}\|_q < 1$, which satisfies*:

(i) $\lambda_1 > d$, $\frac{\lambda_{m+1}}{\lambda_m} > 1 + \mu$, $m = 1, 2, \ldots, M$.
(ii) $m\{x \in I\colon |f(x) - Q(x)| > \delta\} < \delta$.

Choose subsequently $B = B(f, \xi)$ from Lemma 2 and $A = A(q, \epsilon)$ from Lemma 1. Set $\mu = 1/(2K)$. One can prove that if $\xi = \xi(\delta)$ and $\epsilon = \epsilon(B)$ are sufficiently small, then the polynomial

$$Q(x) := \sum b_n e^{i\beta(n)x} A(r_n x)$$

satisfies the requirements of Lemma 3, provided the numbers r_n grow sufficiently fast.

In order to prove Theorem 2 we fix a weight $v(x)$ such that $\widehat{\sqrt{v}}$ is supported by $(-1/2, 1/2)$. This implies Bessel's inequality for the elements of $E(\Lambda)$ in $L^2(w, \mathbb{R})$ for every weight $w \leqslant v$, provided $\gamma(\Lambda) > 1$. Fix also a sequence of functions $f_k \in C(\mathbb{R})$, which is dense in $L^2(v, \mathbb{R})$ and such that f_k vanishes outside of $I_k := (-k, k)$.

Now we define inductively the elements of Λ, polynomials Q_k and sets $E_k \in \mathbb{R}$: On the kth step of induction, suppose that the numbers λ_j, $j \leqslant j(k)$, satisfying (3) are already defined. Set $q = 2 + 1/k$, $\delta = 2^{-k}$. According to Lemma 3, we find the number $\mu(q, \delta, f_k)$. Take $J > j(k)$ so that $\epsilon(J) < \mu$ and continue the sequence $\{\lambda_j\}$ up to $j = J$ keeping the condition (3). Fix $d > (1 + \mu)\lambda_J$ and use Lemma 3 to get a polynomial Q_k and a set $E_k \subset I_k$, $mE_k > mI_k - 2^{-k}$, such that $|f_k(x) - Q_k(x)| \leqslant 2^{-k}$ on E_k. Add the spectrum of Q to Λ. Finally, we obtain an infinite sequence satisfying (3). We may also suppose that it is u.d. and that $\gamma(\Lambda) > 1$.

Define the weight:

$$w(x) := v(x) \inf_k \{\mathbf{1}_{E_k}(x) + \theta(k)\mathbf{1}_{cE_k}(x)\},$$

where the $\theta(k) > 0$ decrease so fast that cE_k contribute $o(1)$ to $\|f_k - Q_k\|_{L^2(w,\mathbb{R})}$. One can check that $w(x) > 0$ a.e. and all conditions of Theorem 2 are satisfied.

2.3. A few words about "time-frequency" localization of generators.

Proposition 1. *As in* [6], *one can construct function g in Theorem* 1 *to be infinitely smooth and even the restriction of an entire function.*

On the other hand, the weight w constructed above is "irregular", so that g cannot decrease fast. This is inevitable, due to the following

Proposition 2. *Let Λ be a u.d. set. If a set* (2) *forms a* (*QF*)*-system in $L^2(\mathbb{R})$ then $\int_{\mathbb{R}} |g| = \infty$.*

Indeed, if $g \in L(\mathbb{R})$, then the corresponding weight $w(x) = |\hat{g}(x)|^2$ is continuous. Hence, one can find a set S of arbitrary large measure which is a finite union of intervals, so that $\inf_S w > 0$. Since $E(\Lambda)$ is a (QF)-system for $L^2(w, \mathbb{R})$, it is so for $L^2(S)$. Then (1) contradicts Theorem 1 in [5].

As a contrast, notice that there exist a u.d. Λ and a function g in the Schwartz class such that (2) is a complete system in $L^2(\mathbb{R})$, see [8].

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