Available online at www.sciencedirect.com

ScienceDirect

C. R. Acad. Sci. Paris, Ser. I 347 (2009) 647–650

COMPTES RENDUS MATHEMATIQUE

Differential Geometry

Minimizers of Kirchhoff's plate functional: Euler–Lagrange equations and regularity

Peter Hornung

Institut für Angewandte Mathematik, Universität Bonn, Endenicher Allee 60, 53115 Bonn, Germany

Received 13 March 2008; accepted after revision 31 March 2009

Available online 21 April 2009

Presented by John Ball

Abstract

Let $S \subset \mathbb{R}^2$ be a bounded C^∞-domain. In this Note we consider $W^{2,2}$ isometric immersions $u : S \to \mathbb{R}^3$ which minimize Kirchhoff's plate functional under boundary conditions prescribing the values of u and of ∇u on parts of ∂S. We derive the Euler–Lagrange equations satisfied by u and we derive regularity results for u. ***To cite this article: P. Hornung, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***
© 2009 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

Résumé

Minimiseurs de la fonctionnelle de Kirchhoff : équations de Euler–Lagrange et régularité. Soit $S \subset \mathbb{R}^2$ un C^∞-domaine borné. Dans cette Note on considère une immersion $W^{2,2}$-isométrique $u : S \to \mathbb{R}^3$ qui minimise la fonctionnelle de Kirchhoff sous les conditions frontières imposant les valeurs de u et ∇u sur des partie de ∂S. On en déduit les équations de Euler–Lagrange satisfaites par u et un résultat de régularité pour u. ***Pour citer cet article : P. Hornung, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***
© 2009 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

1. Introduction

It was recently shown in [2] that the behaviour of thin elastic plates (made of an isotropic material) is ruled by Kirchhoff's energy functional

$$\mathcal{E}(u; S) = \begin{cases} \dfrac{1}{24} \displaystyle\int_S \left|\nabla^2 u(x)\right|^2 \mathrm{d}x & \text{if } u \in W^{2,2}_{\mathrm{iso}}(S; \mathbb{R}^3), \\ +\infty & \text{otherwise.} \end{cases} \tag{1}$$

Here $S \subset \mathbb{R}^2$ is a bounded smooth domain and $W^{2,2}_{\mathrm{iso}}(S; \mathbb{R}^3) = \{u \in W^{2,2}(S; \mathbb{R}^3)\colon\ (\nabla u)^T(\nabla u) = \mathrm{Id}\}$. In this Note we study the regularity of minimizers u of the functional (1) under nontrivial boundary conditions prescribing u and ∇u on a portion $\partial_c S$ of ∂S. Existence is easily established. To obtain regularity information we derive the Euler–Lagrange equations for (1). They are of interest in their own. On the set $W^{2,2}_{\mathrm{iso}}$, the functional (1) agrees (up to a prefactor) with the

E-mail address: peter.hornung@uni-bonn.de.

1631-073X/$ – see front matter © 2009 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.
doi:10.1016/j.crma.2009.03.031

Willmore functional from differential geometry. However, the normal variations commonly used to derive the Euler–Lagrange equations for the Willmore functional fail in our context due to the (nonconvex) isometry constraint. Instead, we use ideas from [9,4] and [3]. In contrast to the Willmore equation, our Euler–Lagrange equations are ordinary differential equations. Roughly speaking, our regularity result states that minimizers of Kirchhoff's functional (under the boundary conditions mentioned earlier) are C^∞ away from three kinds of line segments: Segments which intersect ∂S tangentially, segments which bound regions on which ∇u is locally constant and segments for which $\nabla^2 u$ diverges near one endpoint. At segments of the third kind, we prove that u is precisely C^3 (in the interior), and we obtain sharp estimates for the size of its derivatives. These estimates explain a phenomenon observed in numerical simulations in the physics literature [10]. Moreover, our results clarify some questions raised in [10].

In this Note we give no proofs. Proofs and many more details can be found in [5] and [6]. We refer e.g. to [1] for results on a different constrained Willmore functional.

2. Results

Let $S \subset \mathbb{R}^2$ be a bounded C^∞ domain, let $u_0 \in W^{2,2}_{\mathrm{iso}}(S;\mathbb{R}^3)$ and let $\partial_c S \subset \partial S$ be closed with $\mathcal{H}^1(\partial_c S) > 0$. We set $\mathcal{A}_{u_0}(S, \partial_c S) = \{u \in W^{2,2}_{\mathrm{iso}}(S;\mathbb{R}^3)\colon\ u = u_0 \text{ and } \nabla u = \nabla u_0 \text{ on } \partial_c S\}$. (The values of ∇u on the boundary are understood in the trace sense.) If $u \in \mathcal{A}_{u_0}(S, \partial_c S)$ then clearly $\mathcal{A}_u(S, \partial_c S) = \mathcal{A}_{u_0}(S, \partial_c S)$. The existence of minimizers of $\mathcal{E}(\cdot; S)$ within $\mathcal{A}_{u_0}(S, \partial_c S)$ is an immediate consequence of the weak lower semicontinuity of the $W^{2,2}$-seminorm, see [5]. It can also be established for different kinds of boundary conditions (e.g. prescribing only the values of u).

Let us next recall some notions from [4] and [3]. See [3] for many more details. From now on $S \subset \mathbb{R}^2$ denotes a bounded domain with boundary of class C^∞. For $\mu \in \mathbb{S}^1$ and $x \in S$ we denote by $[x]_\mu$ the connected component of $(x + \operatorname{Span}\mu) \cap S$ which contains x. For $x \in S$ and $\mu \in \mathbb{R}^2 \setminus \{0\}$ we define $\nu(x,\mu) = \inf\{\theta > 0\colon\ x + \theta\mu \notin S\}$. If $\overline{[x]_\mu}$ intersects ∂S transversally, then ν is C^∞ near (x,μ), see [6]. In this case we set $\nu_1(x,\mu)\cdot e_i = \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}(\nu(x + \varepsilon e_i, \mu) - \nu(x,\mu))$.

For given arc-length parametrized $\Gamma \in W^{2,\infty}([-T,T];S)$ and $\kappa_n \in L^2(-T,T)$ set $N = (\Gamma')^\perp$ (the index $\perp$ denotes counter-clockwise rotation by 90 degree), $\kappa = \Gamma''\cdot N$, $s^*(t) = *\nu(\Gamma(t), *N(t))$ (where $* = +, -$), and define $r \in W^{1,2}((-T,T);\mathrm{SO}(3))$ to be the solution to

$$r' = \big((e_1\otimes e_2 - e_2\otimes e_1)\kappa + (e_1\otimes e_3 - e_3\otimes e_1)\kappa_n\big)r \quad \text{and} \quad r(0) = \mathrm{Id}.$$

Set $\gamma' = r^T e_1$, $v = r^T e_2$ and $\gamma(t) = \int_0^t \gamma'$. The mapping $(\Gamma,\kappa_n)\colon [\Gamma(-T,T)] \to \mathbb{R}^3$ is defined by $(\Gamma,\kappa_n)(\Gamma(t) + sN(t)) = \gamma(t) + sv(t)$, where $[\Gamma(-T,T)] = \bigcup\{[\Gamma(t)]_{N(t)}\colon\ t \in (-T,T)\}$ and $s \in (s^-(t), s^+(t))$ for all $t \in (-T,T)$. It is well defined provided that Γ is admissible, i.e. if $[\Gamma(t_1)]_{N(t_1)} \cap [\Gamma(t_2)]_{N(t_2)} \neq \emptyset$ implies $t_1 = t_2$. In what follows we omit the index $N(t)$. The curve Γ is said to be transversal on $[-T,T]$ provided $\overline{[\Gamma(t)]}$ intersects ∂S transversally at both ends for all $t \in [-T,T]$. If this is the case then $S \cap \partial[\Gamma(-T,T)] = [\Gamma(-T)] \cup [\Gamma(T)]$, see [3].

Let $u \in W^{2,2}_{\mathrm{iso}}(S;\mathbb{R}^3)$. We set $C_{\nabla u} = \{x \in S\colon\ \nabla u \text{ is constant in a neighbourhood of } x\}$. This set consists of countably many connected components U. Each U has finite perimeter, and $S \cap \partial U$ is a disjoint union of straight line segments [3]. We denote by $\hat{C}_{\nabla u}$ the union of those U for which $S \cap \partial U$ consists of at least three such segments. Set $D_{\nabla u} = \{x \in S\colon\ \nabla u \text{ is } S\text{-developable in a neighbourhood of } x\}$. By definition, ∇u is S-developable on $X \subset S$ if and only if there exists $q\colon X \to \mathbb{S}^1$ such that ∇u is constant on $[x]_{q(x)}$, and $[x]_{q(x)} \cap [y]_{q(y)} \neq \emptyset$ implies $[x]_{q(x)} = [y]_{q(y)}$ whenever $x, y \in X$. After an appropriate choice of sign, q is locally Lipschitz [7]. We have $\nabla u \in C^0(S;\mathbb{R}^{3\times 2})$ [7,8] and $S \setminus \bar{\hat{C}}_{\nabla u} \subset D_{\nabla u}$ [7,9,3]. For $S_1 \subset D_{\nabla u}$ we set $[S_1]_q = \bigcup\{[x]_{q(x)}\colon\ x \in S_1\}$. In what follows we omit the index q.

If $x \in D_{\nabla u}$ then there is $T > 0$ and $\Gamma\colon [-T,T] \to S$ solving the ODE $\Gamma'(t) = -(q(\Gamma(t)))^\perp$ with $\Gamma(0) = x$, and there is $\kappa_n \in L^2(-T,T)$ such that $u = (\Gamma,\kappa_n)$ on $[\Gamma(-T,T)]$, see [3]. (Here and below, the equality $u = (\Gamma,\kappa_n)$ is understood up to a rigid motion, i.e. there exist $Q \in \mathrm{SO}(3)$ and $d \in \mathbb{R}^3$ such that $d + Qu(x) = (\Gamma,\kappa_n)(x)$ for all $x \in [\Gamma(-T,T)]$.) In this case,

$$\mathcal{E}\big((\Gamma,\kappa_n); \big[\Gamma(-T,T)\big]\big) = \int_{-T}^{T} \kappa_n^2(t)\, g\big(s^\pm(t), \kappa(t)\big)\,\mathrm{d}t, \quad \text{where } g\big(b^\pm, x\big) = \int_{b^-}^{b^+} \frac{1}{1 - sx}\,\mathrm{d}s. \tag{2}$$

If $(\Gamma, \kappa_n) \in W^{2,2}([\Gamma(-T,T)]; \mathbb{R}^3)$ then by (2) necessarily $s^{\pm}\kappa \leqslant 1$ for a.e. $t \in (-T,T)$ and $\kappa_n = 0$ a.e. on the set I_0, which is defined as follows:

$$I_0 := \{t \in [-T,T]:\ s^+(t)\kappa(t) = 1 \text{ or } s^-(t)\kappa(t) = 1\}. \tag{3}$$

Here and below we always refer to the precise representatives of κ and κ_n. For $b^- < 0 < b^+$ and $x \in (\frac{1}{b^-}, \frac{1}{b^+})$ we introduce

$$g_2(b^{\pm}, x) = -\int_{b^-}^{b^+} \frac{1}{(1-sx)^2}\,\mathrm{d}s \quad \text{and} \quad g_3(b^{\pm}, x) = \int_{b^-}^{b^+} \frac{s}{(1-sx)^2}\,\mathrm{d}s.$$

Denote by χ_* the characteristic function of the set where κ has sign $*$. We introduce the functions $\sigma = \sum_* \chi_* s^*$ and $h = \kappa \sum_* *\chi_* \nu_1(\Gamma, *N) \cdot \Gamma'$. We also set

$$F_1 = \sum_* \frac{\nu_1(\Gamma, *N)\cdot \Gamma'}{1 - s^*\kappa} + h g_2(s^{\pm}, \kappa) \quad \text{and} \quad F_2 = \sum_* \frac{s^*\nu_1(\Gamma, *N)\cdot \Gamma'}{1 - s^*\kappa} + \sigma h g_2(s^{\pm}, \kappa).$$

Definition 2.1. A mapping $(\Gamma, \kappa_n) \in W^{2,2}_{\text{iso}}([\Gamma(-T,T)]; \mathbb{R}^3)$ is said to satisfy the Euler–Lagrange equations if Γ is transversal on $[-T,T]$ and if there exist $\lambda_1, \lambda_2 \in \mathbb{R}^3$ and $\lambda_3, \lambda_4 \in \mathbb{R}$ such that the following equations are satisfied for almost every $t \in (-T,T)$:

$$2\big(1 - \chi_{I_0}(t)\big)\kappa_n(t) g\big(s^{\pm}(t), \kappa(t)\big) = -v(t) \cdot \left(\lambda_2 - \lambda_1 \wedge \int_t^T \gamma'\right), \tag{4}$$

$$\big(1 - \chi_{I_0}(t)\big)\kappa_n^2(t) g_2\big(s^{\pm}(t), \kappa(t)\big) = \big(1 - \chi_{I_0}(t)\big)\Omega_2(t), \tag{5}$$

$$\big(1 - \chi_{I_0}(t)\big)\kappa_n^2(t) g_3\big(s^{\pm}(t), \kappa(t)\big) = \Omega_3(t) + \chi_{I_0}(t)\frac{\Omega_2(t)}{\kappa(t)}. \tag{6}$$

Here, I_0 is as defined in (3), and Ω_2 and Ω_3 are the unique Lipschitz continuous solutions to the terminal value problems

$$\Omega_2' = -h\Omega_2 + \kappa_n(\lambda_1 \cdot n) + \kappa_n^2 F_1 \quad \text{and} \quad \Omega_2(T) = \lambda_3 + \lambda_1 \cdot \gamma'(T), \tag{7}$$

$$\Omega_3' = h\sigma\Omega_2 - \kappa_n \gamma' \cdot \left(\lambda_2 - \lambda_1 \wedge \int_t^T \gamma'\right) - \kappa_n^2 F_2 \quad \text{and} \quad \Omega_3(T) = n(T)\cdot\lambda_2 + \lambda_4. \tag{8}$$

In what follows we always suppose that $T > 0$, that $\Gamma \in W^{2,\infty}([-T,T]; S)$ is parametrized by arclength, that Γ is admissible and that $(\Gamma, \kappa_n) \in W^{2,2}_{\text{iso}}([\Gamma(-T,T)]; \mathbb{R}^3)$.

Theorem 2.2. *If (Γ, κ_n) minimizes $\mathcal{E}(\cdot; [\Gamma(-T,T)])$ within $\mathcal{A}_{(\Gamma,\kappa_n)}([\Gamma(-T,T)], [\Gamma(-T)] \cup [\Gamma(T)])$ and if Γ is transversal on $[-T,T]$, then (Γ, κ_n) solves the Euler–Lagrange equations in the sense of Definition* 2.1.

Roughly speaking, the Euler–Lagrange equations are obtained by varying the curvatures κ and κ_n. Eq. (4) arises from variations of κ_n, and (6) from variations of κ. Eq. (5) is related to (4), (6) and arises from a third, more implicit kind of variations. The multipliers λ_i arise from the boundary conditions, which can be reformulated as integral constraints on κ and κ_n. The relevance of Theorem 2.2 is this: As seen above, if $x \in D_{\nabla u}$ then there is $T > 0$ and (Γ, κ_n) such that $u = (\Gamma, \kappa_n)$ on $[\Gamma(-T,T)]$. If, moreover, $x \notin \Sigma_\tau \cup \Sigma_c$ (as defined below) then (Γ, κ_n) satisfies the hypotheses of Theorem 2.2 for small $T > 0$ [6].

We say that (Γ, κ_n) satisfies condition (A) if the following holds: If J is a nondegenerate maximal interval in $\{t \in [-T,T]:\ \kappa_n(t) = 0\}$ with $\mathcal{L}^1(J \cap I_0) = 0$ then κ is not constant on J.

Theorem 2.3. *Assume that* (Γ, κ_n) *solves the Euler–Lagrange equations in the sense of Definition* 2.1 *and that* $\mathcal{L}^1(\{t \in (-T, T): \kappa_n(t) \neq 0\}) > 0$. *Then*

$$\kappa_n \in C^0\big([-T, T]\big) \quad \text{and} \quad \kappa, \kappa_n \in C^\infty\big([-T, T] \setminus \partial\big\{t \in [-T, T]: \kappa_n(t) = 0\big\}\big),$$

and the set I_0 *has empty interior.*

Assume, in addition, that condition (A) *is satisfied. Then the following hold*: κ_n *has finitely many zeros and it changes its sign at each of them. Moreover,* $\kappa \in C^\infty([-T, T] \setminus I_0) \cap C^2([-T, T])$ *and* $\kappa_n \in C^\infty([-T, T] \setminus I_0) \cap C^1([-T, T])$. *Finally, if there exists a* $\delta > 0$ *such that* $\kappa \in C^{2,\delta}(-T, T)$ *or* $\kappa_n \in C^{1,\delta}(-T, T)$ *then* $I_0 = \emptyset$.

Remark 1. (i) If $\kappa_n = 0$ almost everywhere on $(-T, T)$ then (Γ, κ_n) is affine on $[\Gamma(-T, T)]$, so the assumption $\mathcal{L}^1(\{\kappa_n \neq 0\}) > 0$ is not restrictive. It turns out that, on the level of surfaces, hypothesis (A) is not restrictive either, see [6].

(ii) Near I_0 we obtain the following estimates (see [6]): Suppose that $t_0 \in I_0$, denote by $* \in \{+, -\}$ the sign of $\kappa(t_0)$ and set $\alpha^* = 1 - s^*\kappa$. Then, for t near t_0, we have: $\sqrt{\alpha^*(t)}\,|\log \alpha^*(t)| \sim |t - t_0|$, $|\kappa - \kappa(t_0)| \sim \alpha^*$, $|\kappa'| \sim \frac{\sqrt{\alpha^*}}{|\log \alpha^*|}$, $|\kappa_n| \sim \sqrt{\alpha^*}$, $|\kappa_n'| \sim |\log \alpha^*|^{-1}$, and $|\kappa''| \leqslant C(\log \alpha^*)^{-2}$. In particular, κ', κ'' and κ_n' are continuous and zero at t_0. Moreover, $\nabla^2(\Gamma, \kappa_n)$ diverges near $\Gamma(t_0) + s^*(t_0)N(t_0) \in \partial S$.

(iii) Since $\kappa_n = 0$ on I_0, Theorem 2.3 shows that I_0 is finite. This is a key regularity result which contrasts with the example in the appendix to [3]. There it is shown that for arbitrary $W^{2,2}$ isometries the set I_0 can be open and dense. The last statement of Theorem 2.3 shows that the regularity found for κ and κ_n is optimal. It could only be improved by showing that $I_0 = \emptyset$. Numerical simulations in [10] strongly suggest that in general $I_0 \neq \emptyset$.

To state our main result in terms of the surface u, we introduce three kinds of line segments:

$$\Sigma_\tau = \big\{x \in D_{\nabla u}: \overline{[x]} \text{ intersects } \partial S \text{ tangentially}\big\},$$
$$\Sigma_0 = \big\{x \in D_{\nabla u} \setminus \Sigma_\tau: \text{ there is a } \nabla u\text{-integral curve } \Gamma \text{ and } t_0 \in I_0 \text{ such that } [x] = \big[\Gamma(t_0)\big]_{N(t_0)}\big\},$$
$$\Sigma_c = \text{closure of } \big\{x \in D_{\nabla u}: \overline{[x]} \text{ intersects } \partial_c S\big\}.$$

Theorem 2.4. *Let* u *be a minimizer of* $\mathcal{E}(\cdot; S)$ *within the class* $\mathcal{A}_u(S, \partial_c S)$. *Then*

$$u \in C^3\big(S \setminus \big(\partial\Sigma_\tau \cup \Sigma_c \cup (\partial D_{\nabla u} \cap \partial\hat{C}_{\nabla u})\big); \mathbb{R}^3\big) \cap C^\infty\big(S \setminus \big(\Sigma_0 \cup \partial\Sigma_\tau \cup \Sigma_c \cup (\partial D_{\nabla u} \cap \partial\hat{C}_{\nabla u})\big); \mathbb{R}^3\big).$$

Remark 2. (i) Σ_0 consists of countably many line segments which can accumulate only at $\partial S \cup \Sigma_c \cup \Sigma_\tau$ [6]. If $x \in \Sigma_0$ then $u(x)$ is a planar point on the surface $u(S)$ by the observation following (2).

(ii) On Σ_c the mapping u is determined by its boundary conditions on $\partial_c S$. If S is convex then Σ_τ is empty. Under certain conditions on $\partial_c S$, the set $\partial\hat{C}_{\nabla u}$ consists of only finitely many segments, see [6].

(iii) By Theorem 2.3, minimizers are not better than C^3 at Σ_0.

Acknowledgements

Preliminary versions of some of the results were obtained in my PhD thesis. I thank my advisor Stefan Müller for proposing this problem and for helpful suggestions.

References

[1] C. Bohle, G.P. Peters, U. Pinkall, Constrained Willmore surfaces, Calc. Var. 32 (2008) 263–277.
[2] G. Friesecke, R. James, S. Müller, A theorem on geometric rigidity and the derivation of nonlinear plate theory from three-dimensional elasticity, Comm. Pure Appl. Math. 55 (2002) 1461–1506.
[3] P. Hornung, A density result for $W^{2,2}$ isometric immersions, MIS MPG Preprint, 2007.
[4] P. Hornung, Approximating $W^{2,2}$ isometric immersions, C. R. Acad. Sci. Paris, Ser. I 346 (2008) 189–192.
[5] P. Hornung, Flat minimizers of the Willmore functional: Euler–Lagrange equations, Preprint, Universität Bonn, 2008.
[6] P. Hornung, Regularity results for flat minimizers of the Willmore functional, Preprint, Universität Bonn, 2008.
[7] B. Kirchheim, Geometry and Rigidity of Microstructures, Habilitation thesis, University of Leipzig, 2001.
[8] S. Müller, M.R. Pakzad, Regularity properties of isometric immersions, Math. Z. 251 (2005) 313–331.
[9] M.R. Pakzad, On the Sobolev space of isometric immersions, J. Differential Geom. 66 (1) (2004) 47–69.
[10] E.L. Starostin, G.H.M. van der Heijden, The shape of a Möbius strip, Nature Materials 6 (2007) 563–567.