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Differential Geometry

The Witten complex for algebraic curves with cone-like singularities

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Abstract

The Witten deformation is an analytical method proposed by Witten which, given a function $f : M \to \mathbb{R}$ on a smooth compact Riemannian manifold M, leads to a proof of the Morse inequalities. In this Note we generalise the Witten deformation to singular complex algebraic curves X with cone-like singularities, and functions on X which we call admissible Morse functions. They are particular examples of stratified Morse functions in the sense of the theory developed by Goresky/MacPherson. ***To cite this article: U. Ludwig, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Le complexe de Witten sur des courbes algébriques complexes à singularités coniques. Soit M une variété Riemannienne compacte et soit $f : M \to \mathbb{R}$ une fonction de Morse sur M. La méthode de Witten utilise une déformation du complexe de de Rham pour démontrer les inegalités de Morse. Le but de cette Note est d'étendre cette méthode au cas des courbes algébriques complexes à singularités coniques, munis de fonctions appelées fonctions de Morse admissibles. Ces fonctions sont des fonctions de Morse stratifiées au sens de la théorie de Goresky/MacPherson. ***Pour citer cet article : U. Ludwig, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

1. Introduction

Let X be a singular complex algebraic curve. For simplicity of presentation we will assume that all singularities $\Sigma := \{p_1, \dots, p_N\}$ of X are unibranched. For $i = 1, \dots, N$ we denote by $m(p_i) = m_i \in \mathbb{N}$, $m_i \geqslant 2$, the multiplicity of X at p_i. Let g be a metric on X such for each $p_i \in \Sigma$ there exists an open neighbourhood $U(p_i)$ in X such that $(U(p_i) - p_i, g_{|U_i - p_i})$ is isometric to $(\mathrm{cone}_\epsilon(S^1_{m_i}), \mathrm{d}r^2 + r^2\,\mathrm{d}\varphi^2)$ for some $\epsilon > 0$. Hereby for $m \in \mathbb{N}$ we denote by S^1_m the circle of length $2\pi m$ and by $\mathrm{cone}_\epsilon(S^1_m) := \{(r, \varphi) \mid r \in (0, \epsilon),\ \varphi \in S^1_m\}$. We study a certain class of functions on X, which we call admissible Morse functions:

Definition 1. Let $f : X \to \mathbb{R}$ be a continuous function which is smooth outside the singularities of X. The function f is called an admissible Morse function if the restriction $f_{|X-\Sigma}$ is Morse (in the smooth sense) and moreover if for

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any singular point $p \in \Sigma$ there exist $a_p, b_p \in \mathbb{R}$, $(a_p, b_p) \neq (0,0)$, such that the function f has the following form in local coordinates (r,φ) near p: $f(r,\varphi) = f(p) + r(a_p \cos(\varphi) + b_p \sin(\varphi))$.

The above assumptions on g and f are motivated by the following observation: (X, g) is a metric model for the singular complex algebraic curve $X \subset \mathbb{P}^n(\mathbb{C})$, equipped with the metric $\tilde{g}$ induced by the Fubini–Study metric on $\mathbb{P}^n(\mathbb{C})$. Admissible Morse functions are particular examples of stratified Morse functions in the sense of the theory developed by Goresky/MacPherson in [4]. More precisely, let $p \in X$ be a unibranched singular point of multiplicity m. One has a normalisation map $\pi : V(0) \subset \mathbb{C} \to U(p) \cap X,\ t \mapsto (z_1(t), \dots, z_{n+1}(t)) = (t^m, t^{q_2} f_2(t), \dots, t^{q_{n+1}} f_{n+1}(t))$ such that $\pi_{|V-\{0\}}$ is a biholomorphic map, $m < q_2 < q_3 < \cdots < q_{n+1}$ and $f_k(0) \neq 0$. Then $(V - \{0\}, \pi^*\tilde{g})$ is isometric to $(\mathrm{cone}(S^1_m), (1 + \mathrm{O}(r^{1/m}))(\mathrm{d}r^2 + r^2\,\mathrm{d}\varphi^2))$. The affine line $l := \{z_2 = \cdots = z_n = 0\}$ is the tangent line to X. Let F: $\mathbb{P}^n(\mathbb{C}) \cap U(p) \to \mathbb{C}$ be a holomorphic function such that $f := \mathrm{Re}(F)_{|X} : X \cap U(p) \to \mathbb{R}$ is a stratified Morse function in the sense of [4] (Part II). The non-degeneracy condition in [4] implies that locally near p the function F has the form $F = F(p) + \sum a_i z_i + \mathrm{O}(z^2)$, where $a_1 \neq 0$. In local coordinates (r,φ) we get $f = r(a\cos\varphi + b\sin\varphi) + \mathrm{O}(r^{1+\delta})$, $\delta > 0$.

The de Rham complex of smooth forms with compact support $(\Omega_0^*(X - \Sigma), d, \langle,\rangle)$ has a unique extension into a Hilbert complex $(\mathcal{C}, \bar{d}, \langle,\rangle)$ in the Hilbert space of square integrable forms (see [3] for the definition of a Hilbert complex). Hereby $\bar{d}$ denotes the closure of d and $\langle,\rangle$ denotes the L^2-metric: $\langle \alpha, \beta\rangle = \int_{X-\Sigma} \alpha \wedge *_g \beta$. The cohomology of the complex $(\mathcal{C}, \bar{d}, \langle,\rangle)$ is the so-called L^2-cohomology of X, $H^i_{(2)}(X) := \ker \bar{d}_i / \operatorname{im} \bar{d}_{i-1}$, $i = 0,1,2$. We denote by $b_i^{(2)}(X) := \dim H^i_{(2)}(X)$, the L^2-Betti numbers of X. The index of a critical point $p \in X - \Sigma$ is the number of negative eigenvalues of the Hessian of f in p. The singular points $p \in \Sigma$ are considered to be critical points for f of index 1. For $i = 0,1,2$ we denote by $\mathrm{Crit}_i(f)$ the set of critical points for f of index i and by $c_i(f)$ the number of critical points of index i "counted with multiplicities": $c_i(f) := \sum_{p \in \mathrm{Crit}_i(f)} n(p)$, where $n(p) = 1$ for all critical points $p \in X - \Sigma$ and $n(p) = m(p) - 1$ for $p \in \Sigma$.

The main goal of this Note is to generalise the Witten deformation (see [7]) to the above situation and to give an analytic proof of the below Morse inequalities:

Theorem 2. *Let (X, g) be a complex algebraic curve as above and let $f : X \to \mathbb{R}$ be an admissible Morse function on X. Then the following Morse inequalities hold*:

$$c_0(f) \geqslant b_0^{(2)}(X), \quad c_1(f) - c_0(f) \geqslant b_1^{(2)}(X) - b_0^{(2)}(X), \quad \sum_{i=0}^{2} (-1)^i c_i(f) = \sum_{i=0}^{2} (-1)^i b_i^{(2)}(X). \tag{1}$$

Note that the local situation near singular points of X is quite different from the one near smooth critical points. The contribution of the singular points to the Morse inequalities is closely related to the lack of essential self-adjointness of the Laplace operator (acting on smooth forms with compact support) in the presence of singularities. Moreover the singular points of X contribute to the Morse inequalities in degree 1 and have to be counted with "multiplicities".

Note that from Theorem 2 one can recover the Morse inequalities for intersection homology with middle perversity (for the singular complex curve X with stratified Morse function f) known already from stratified Morse theory [4]: by the Cheeger–Goresky–MacPherson conjecture (proved in [6] in arbitrary dimension) the intersection cohomology of the singular complex algebraic curve X is isomorphic to the L^2-cohomology of X with respect to the metric on X induced by the Fubini–Study metric on $\mathbb{P}^n(\mathbb{C})$. Since the L^2-cohomology is a quasi-isometry invariant, the intersection cohomology of X is also isomorphic to the L^2-cohomology of X with respect to the conic metric g. Extending Wittens program to the higher dimensional case will be more challenging, given that the natural metrics on algebraic varieties, i.e. those induced from a metric on projective space are in general not of cone-type.

2. The Witten deformation of the complex $(\mathcal{C}, \bar{d}, \langle,\rangle)$ and the spectral gap theorem

The Witten method consists in deforming the de Rham complex into a complex $(\Omega_0^*(X - \Sigma), d_t, \langle,\rangle)$, where $d_t\omega = e^{-tf}\,\mathrm{d}(e^{tf}\omega) = \mathrm{d}\omega + t\,\mathrm{d}f \wedge \omega$. We denote by δ_t the formal adjoint of d_t with respect to the L^2-metric $\langle,\rangle$.

Proposition 3. *The complex* $(\Omega_0^*(X-\Sigma), d_t, \langle,\rangle)$ *has a unique extension into a Hilbert complex* $(\mathcal{C}, \bar{d}_t, \langle,\rangle)$. *Moreover the associated Laplacian* $\Delta_t = \bar{d}_t\bar{\delta}_t + \bar{\delta}_t\bar{d}_t$ *with* $\mathrm{dom}(\Delta_t) = \{\Psi \in \mathrm{L}^2(\Lambda^*(T^*(X-\Sigma))) \mid \bar{d}_t\Psi, \bar{\delta}_t\Psi, \bar{d}_t\bar{\delta}_t\Psi, \bar{\delta}_t\bar{d}_t\Psi \in \mathrm{L}^2(\Lambda^*(T^*(X-\Sigma)))\}$ *is a nonnegative, self-adjoint, discrete operator such that* $\ker(\Delta_t) \simeq \ker \bar{d}_{t,i}/\operatorname{im}\bar{d}_{t,i-1} \simeq H^i_{(2)}(X)$.

We call the operator Δ_t with $\mathrm{dom}(\Delta_t)$ the Witten Laplacian. The following result on the spectral properties of the Witten Laplacian is the key result to the proof of Theorem 2.

Theorem 4 *(Spectral gap theorem). There exist constants* C_1, C_2, C_3 *and* $t_0 > 0$ *depending on* X *and* f *such that* $\operatorname{spec}\Delta_{t,i} \cap (C_1 e^{-C_2 t}, C_3 t) = \emptyset$ *for any* $t > t_0$.

The proof of the spectral gap theorem consists in two steps. First we develop a model operator for Δ_t in the neighbourhood of a singular point $p \in \Sigma$ of X of multiplicity m. A simple computation using (1) shows that locally near p we have $\Delta_t = \Delta + (a_p^2 + b_p^2)t^2$. We therefore define the model Witten Laplacian $\mathbf{\Delta}_t^p$ as the following self-adjoint operator acting on forms on the infinite cone $\mathrm{cone}(S_m^1)$:

$$\mathbf{\Delta}_t^p = \Delta + \left(a_p^2 + b_p^2\right)t^2 \quad \text{and} \quad \mathrm{dom}\left(\mathbf{\Delta}_t^p\right) = \left\{\Psi \mid \Psi, \bar{d}_t\Psi, \bar{\delta}_t\Psi, \bar{d}_t\bar{\delta}_t\Psi, \bar{\delta}_t\bar{d}_t\Psi \in \mathrm{L}^2\left(\Lambda^*\left(T^*\left(\mathrm{cone}\left(S_m^1\right)\right)\right)\right)\right\}.$$

We can show the following "local spectral gap theorem":

Proposition 5. $\operatorname{spec}(\mathbf{\Delta}_{t,i}^p) = [t'^2, \infty)$ *in case* $i = 0, 2$. *Moreover* $\operatorname{spec}(\mathbf{\Delta}_{t,1}^p) = \{0\} \cup [t'^2, \infty)$ *with* $\dim\ker(\mathbf{\Delta}_{t,1}^p) = m-1$, *and all forms* $\omega \in \ker(\mathbf{\Delta}_{t,1}^p)$ *have the following asymptotic behaviour*: $\omega \approx \frac{e^{-t'r}}{\sqrt{t'r}}\,\mathrm{d}r + \sqrt{t'r}\,e^{-t'r}\,\mathrm{d}\varphi$ *for* $t'r \gg 0$, *where* $t' = \sqrt{a_p^2 + b_p^2}\cdot t$.

In the second step of the proof of the spectral gap theorem we follow the proof in the smooth case (see [1], Section 9). Recall from the smooth theory that the model Witten Laplacian $\mathbf{\Delta}_t^p$ in the neighbourhood of a critical point $p \in X - \Sigma$ has discrete spectrum $\operatorname{spec}(\mathbf{\Delta}_t^p) = 2\mathbb{N}t$ and $\dim\ker(\mathbf{\Delta}_t^p) = 1$. We denote by $\omega^p(t)$ the generator of $\ker(\mathbf{\Delta}_t^p)$. For a singular point $p \in \Sigma$ of multiplicity m we denote by $\{\omega_j^p(t) \mid j = 1, \dots, m-1\}$ a basis of $\ker(\mathbf{\Delta}_t^p)$. Let $\nu_\epsilon : \mathbb{R}^+ \to \mathbb{R}$ be a cut-off function with $\nu_\epsilon = 1$ in $[0, \epsilon/2]$. The forms $\Phi_j^p(t) := \nu_\epsilon(\|x\|)\omega_j^p(t)$ can be identified with L^2-forms on X. We denote by

$$E(t) := \operatorname{span}\left\{\left\{\Phi_1^p(t) := \nu_\epsilon\omega^p(t) \mid p \in \operatorname{Crit}(f) - \Sigma\right\} \cup \left\{\Phi_j^p(t) \mid p \in \Sigma,\ j \in I_p := \left\{1, \dots, m(p)-1\right\}\right\}\right\}.$$

We get an orthogonal splitting $\mathrm{L}^2(\Lambda^*(T^*(X-\Sigma))) = E(t) \oplus E(t)^\perp$. The closed operator $A_t := \bar{d}_t + \bar{\delta}_t$ with $\mathrm{dom}(A_t) = \mathrm{dom}(\bar{d}_t) \cap \mathrm{dom}(\bar{\delta}_t) \subset \mathrm{L}^2(\Lambda^*(T^*(X-\Sigma)))$ can be written in matrix form

$$A_t = \begin{pmatrix} A_{t,1} & A_{t,2} \\ A_{t,3} & A_{t,4} \end{pmatrix} \quad \text{according to the splitting } E(t) \oplus E(t)^\perp.$$

Note that $\mathrm{dom}(A_t)$ equipped with the norm $\|s\|_1 := \sqrt{\|(d+\delta)s\|^2 + \|s\|^2}$ is complete. We show the following estimates on A_t as $t \to \infty$:

Proposition 6. *There exist constants* $c, C > 0$ *and* $t_0 > 0$ *such that for all* $t > t_0$ *we have*

(i) *For all* $s \in E(t)$ *we have* $\|A_t s\| = \mathrm{O}(e^{-ct})\|s\|$. *In particular* $\|A_{t,1}s\| = \mathrm{O}(e^{-ct})\|s\|$, $\|A_{t,3}s\| = \mathrm{O}(e^{-ct})\|s\|$.
(ii) *For all* $s \in E(t)^\perp \cap \mathrm{dom}(A_t)$ *we get*: $\|A_{t,2}s\| \leqslant \mathrm{O}(e^{-ct})\|s\|$, $\|A_{t,4}s\| \geqslant C(\|s\|_1 + \sqrt{t}\|s\|)$.

The proof of Proposition 6 is similar to the corresponding statements in the smooth case (see [1], Section 9). To prove the estimates for forms s with support in a neighbourhood of a singular point of X Proposition 5 on the spectrum of the model Witten Laplacian is crucial. As in [1], Section 9 (c) and (e), Proposition 6 allows to give estimates for the resolvent of $A_t - \lambda : \mathrm{dom}(A_t) \to \mathrm{L}^2(\Lambda^*(T^*(X-\Sigma)))$, where $\lambda \in \mathbb{C}$, $|\lambda| \in [e^{-ct/2}, \frac{C\sqrt{t}}{2}]$, with constants c, C as in Proposition 6. We deduce the invertibility of the operator $A_t - \lambda$, and since $\Delta_t - \lambda^2 = (A_t - \lambda)(A_t + \lambda)$ we thus get Theorem 4.

3. Proof of the Morse inequalities

For $i = 0, 1, 2$ we define the $\mathbb{R}$-vector space C_i by $C_i := \bigoplus_{p \in \mathrm{Crit}_i(f) \setminus \Sigma} \mathbb{R} \cdot e_1^p \oplus \bigoplus_{p \in \mathrm{Crit}_i(f) \cap \Sigma, j \in I_p} \mathbb{R} \cdot e_j^p$. We define a linear map $J_i(t) : C_i \to \mathcal{C}_{t,i}$ by $J_i(t)(e_j^p) = \Phi_j^p(t)$. We denote by $(\mathbb{F}_t, \overline{d_t}, \langle, \rangle)$ the subcomplex of $(\mathcal{C}_t, \overline{d_t}, \langle, \rangle)$ generated by the eigenforms of Δ_t to eigenvalues lying in $[0, 1]$. We denote moreover by $P(t, [0, 1])$ the orthogonal projection operator from $\mathcal{C}_t$ on $\mathbb{F}_t$ with respect to $\langle, \rangle$.

Proposition 7. *There exist a constant $c > 0$ and an L^2-integrable function $\rho : X \to \mathbb{R}$ such that for all $v \in C_i$ and all $x \in X$:* $|[(P_i(t, [0, 1]) \circ J_i(t) - J_i(t))v](x)| = \rho(x) \mathrm{O}(e^{-ct}) \|v\|$.

The proof of Proposition 7 is similar to the proof of Theorem 6.7 in [2], where we have to replace the elliptic estimates with the so-called singular elliptic estimates (see [5]). We can now prove the Morse inequalities: As a corollary of Proposition 7 we get that the linear map $P_i(t, [0, 1]) \circ J_i(t) : C_i \to \mathbb{F}_{t,i}$ is an isomorphism from C_i onto $\mathbb{F}_{t,i}$. Therefore the complex $(\mathbb{F}_t, \overline{d_t}, \langle, \rangle)$ is a finite dimensional subcomplex of $(\mathcal{C}_t, \overline{d_t}, \langle, \rangle)$ with $\dim \mathbb{F}_{t,i} = c_i(f)$. By Proposition 3 moreover $\ker(\Delta_t) \simeq H^*(\mathbb{F}_t, \overline{d_t}, \langle, \rangle) \simeq H^*_{(2)}(X)$. The Morse inequalities in Theorem 2 now follow by a standard algebraic argument.

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References

[1] J.-M. Bismut, G. Lebeau, Complex immersions and Quillen metrics, Publ. Math. Inst. Hautes Etud. Sci. 74 (1991) 1–297.
[2] J.-M. Bismut, W. Zhang, Milnor and Ray–Singer metrics on the equivariant determinant of a flat vector bundle, Geom. Funct. Anal. 4 (2) (1994) 136–212.
[3] J. Brüning, M. Lesch, Hilbert complexes, J. Funct. Anal. 108 (1) (1992) 88–132.
[4] M. Goresky, R. MacPherson, Stratified Morse Theory, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) (Results in Mathematics and Related Areas (3)), vol. 14, Springer-Verlag, Berlin, 1988.
[5] M. Lesch, Operators of Fuchs Type, Conical Singularities, and Asymptotic Methods, Teubner-Texte zur Mathematik, vol. 136, Stuttgart, 1997.
[6] T. Ohsawa, On the L^2-cohomology of complex spaces, Math. Z. 209 (4) (1992) 519–530.
[7] E. Witten, Supersymmetry and Morse theory, J. Differential Geom. 17 (4) (1982) 661–692.

Further reading

[1] B. Helffer, J. Sjöstrand, Puits multiples en mécanique semi-classique. IV : Étude du complexe de Witten, Comm. Partial Differential Equations 10 (3) (1985) 245–340.